Generalized Morse Theory for tubular neighborhoods

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Abstract

We define a notion of Morse function and establish Morse Theory-like theorems over a class of compact subsets of Euclidean spaces verifying weak regularity assumptions. Our approach involves non-smooth analysis over Lipschitz functions and the μ-reach of a set, which is a quantity used in geometric inference to study non-smooth, non convex subsets of a Euclidean space. This paper adds to previous works that were able to define Morse functions for several classes of subsets of Euclidean spaces such as submanifolds, Whitney-Stratified sets and sets with positive reach. Our conditions are the positiveness of the μ-reach of our sets and of the reach of their complement sets, as well as the full-dimensionality of their tangent cones. In particular, we prove that this class is vast among tubular neighborhoods as it notably contains all but a finite number of offsets of any subanalytic sets, or any small offset of a compact set with positive μ-reach.

1 Introduction

In his celebrated book *Morse Theory* [1], Milnor describes how the changes in topology of the closed sublevel sets $X_c \coloneqq f^{-1}(-\infty, c]$ when c increases can be derived from $f : X \to \mathbb{R}$ when X is a compact C^2 manifold and f is smooth and sufficiently generic. Such generic functions are called *Morse Functions*. In this setting, Milnor shows that topological changes only happen around a finite number of values called *critical values* determined by the values the function f takes at the *critical points*, which are the points where the differential of f vanishes. Around a critical point x with critical value $c = f(x)$, the topology of the sublevel sets $X_{c+\varepsilon}$ is obtained from $X_{c-\varepsilon}$ by gluing a cell around x when ε is small enough.

A smooth function $f : X \to \mathbb{R}$ is said to be Morse when its Hessian is non-degenerate at every critical point. In this case the previous considerations can be summarized by the two fundamental results of Morse Theory, which we call *Morse Theorems*:

- Let $a < b \in \mathbb{R}$. If $[a, b]$ does not contain any critical value of f, X_a has the same homotopy type as X_b. This is the *Constant homotopy type Lemma*.
- Around a critical value c of f, the homotopy type of $X_{c+\varepsilon}$ is obtained from $X_{c-\varepsilon}$ by gluing a λ_i cell around each critical point $x_i \in f^{-1}(c)$, when ε is small enough. This is the *Handle attachment Lemma*.

Morse functions are plentiful: for any manifold X embedded in a Euclidean space $\mathbb{R}^d$, the functions $d_{\{x\}|X}$ is Morse for $\mathcal{H}^d$-almost all x, and the height functions $x \mapsto \langle x, v\rangle$ restricted to X are Morse for $\mathcal{H}^{d-1}$-almost all $v \in \mathbb{S}^{d-1}$ [1]. Moreover, "almost all" functions are Morse in the Whitney topology on X, as the set they form is dense and open.

The aim of this article is to extend the class of sets X and the definition of Morse functions for which the Morse theorems stand. A recent work from Monod, Song, Kim [2] showed that for a generic surface $S \subset \mathbb{R}^3$, the Morse Theorems are verified for $f = d_S$ the distance to S when X is a submanifold.

When f is smooth, Fu [3] narrowed the assumptions to any set X with $C^{1,1}$ boundary and more generally to sets with positive reach. His reasoning is the main inspiration for the present article, as we adapt his proofs using non-smooth analysis to different regularity assumptions on X, namely requiring that X has a positive μ-reach for some $\mu \in (0,1]$, that its complement set $^\neg X := \overline{\mathbb{R}^d \setminus X}$ has a positive reach, and that its tangent cones have full-dimensionality. All these notions are described in Section 2.1. In particular the class of compact sets of $\mathbb{R}^d$ verifying all those assumptions encompasses almost any tubular neighborhood of subanalytic sets or small offsets of sets with positive μ-reach.

Here is the major result of this paper formulated informally.

Theorem 1.1: Informal Generalized Morse Theory

Let $X \subset \mathbb{R}^d$ be such that $\mathrm{reach}_\mu(X) > 0$ and $\mathrm{reach}(^\neg X) > 0$ for a certain $\mu \in (0,1]$. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $f_{|X}$ admits only non degenerate critical points.

Then for every regular value c of $f_{|X}$, $X_c := X \cap f^{-1}(-\infty, c]$ has the homotopy type of a CW-complex with extra cells added independently in the sense of persistence theory at critical values, whose dimension depend explicitly on the curvatures of X.

Outline

In Section 2 we define the objects used throughout this article.

- In Section 2.1 we define and illustrate the basic tools of our study. This includes the reach_μ and the reach of a compact subset of $\mathbb{R}^d$, Clarke gradients of locally lipschitz functions,normal and tangent cones of an object with positive reach.
- In Section 2.2 we define the unit Normal Bundle of a set with positive reach and derive some of its geometric properties.
- Section 2.3 recalls definitions and notations of critical points and Hessian for a restricted function $f_{|X}$ for sets with positive reach from Fu [3].
- Section 2.4 focuses on properties of locally Lipschitz functions for non-smooth analysis. We build a retraction between sublevel sets of such functions assuming their Clarke gradient stays away from zero.
- In Section 2.5 we establish a fundamental link between the Normal Bundle of a set X and the Clarke gradient of its distance function d_X. This crucial step allows us to use results from non-smooth analysis on assumptions about critical points of $f_{|X}$.

Section 3 articulates the previous results to establish the main theorem.

- In Section 3.1 we describe the regularity conditions we impose on X to prove Morse Theory results.
- In Section 3.2 we describe how to build a function $f_{r,c}$ such that $(X^{-r}, f_{r,c})$ are smooth surrogates for (X, f) in the sense that the smooth sublevel sets X_c^{-r} and X_c have the same homotopy type when c is a regular value and r is small enough. To that end we consider some locally Lipschitz functions and prove that they verify the assumptions needed in the theorems of Section 2.4. The retractions obtained are used to build a homotopy equivalence.
- In Section 3.3 we show that in between critical values, the topology of sublevel sets stays constant. This is done by applying Section 2.4 using computations from the previous section.

- Section 3.4 describes the topological changes happening around a critical value as long as it has only one corresponding critical point which is non-degenerate. We adapt the proof from Fu [3] to our setting, circumventing the problem of considering sets with reach 0 using non-smooth analysis.

- Section 3.5 describes topological changes around a critical value admitting several critical points that are all non-degenerate.

2 Definitions and useful lemmas

2.1 Preliminaries

- Throughout this paper, the *complement set* of a closed set $X \subset \mathbb{R}^d$ will denote $\neg X = \overline{\mathbb{R}^d \setminus X}$ the closure of the classical complement set.

- Let A be a subset of $\mathbb{R}^d$. It *distance function* is $d_A : x \mapsto \inf\{||x-a|| \mid a \in A\}$. Any such function is 1-Lipschitz and thus differentiable almost everywhere. For any positive r and X subset of $\mathbb{R}^d$, define the r and $-r$ tubular neighborhoods of X (see Figure 2) as follows:

$$\begin{aligned} X^r &:= \{x \in \mathbb{R}^d \mid d_X(x) \leq r\} \\ X^{-r} &:= \{x \in \mathbb{R}^d \mid d_{\neg X}(x) \geq r\} \end{aligned}$$

  The *Hausdorff Distance* between two subsets A, B of $\mathbb{R}^d$ is the infimum of the $t \in \mathbb{R}$ such that $B \subset A^t$ and $A \subset B^t$. This distance yields a topology on the set of compact subsets of $\mathbb{R}^d$.

- A *Cone* A in $\mathbb{R}^d$ is a set stable under multiplication by a positive number, i.e for all $\lambda > 0$, we have $\lambda A \subset A$. Given any $B \subset \mathbb{R}^d$, denote $\mathrm{Cone}\, B$ the smallest cone containing B, defined as the image of $[0, \infty) \times B$ by the map $(\lambda, x) \mapsto \lambda x$. In the same vein, denote $\mathrm{Conv}\, B$ the *convex hull* of B to be the smallest convex set containing B, consisting in all convex combinations of elements of B. A *Convex cone* is a subset of $\mathbb{R}^d$ which is both a cone and convex. The dimension of a cone or a convex set is the dimension of the vector space it spans. Given any set $B \subset \mathbb{R}^d$, its *polar cone* or *dual cone* B^{o} is the convex cone of $\mathbb{R}^d$ defined by :

$$B^{\mathrm{o}} = \{u \in \mathbb{R}^d \mid \langle u, b\rangle \leq 0 \quad \forall\, b \in B\}.$$

  The polar cone operation is idempotent on convex cones, as it notably verifies the following identity :

$$(B^{\mathrm{o}})^{\mathrm{o}} = \mathrm{Conv}\,(\mathrm{Cone}\, B)\,.$$

- Given a subset X of $\mathbb{R}^d$, define its *distance to 0* as

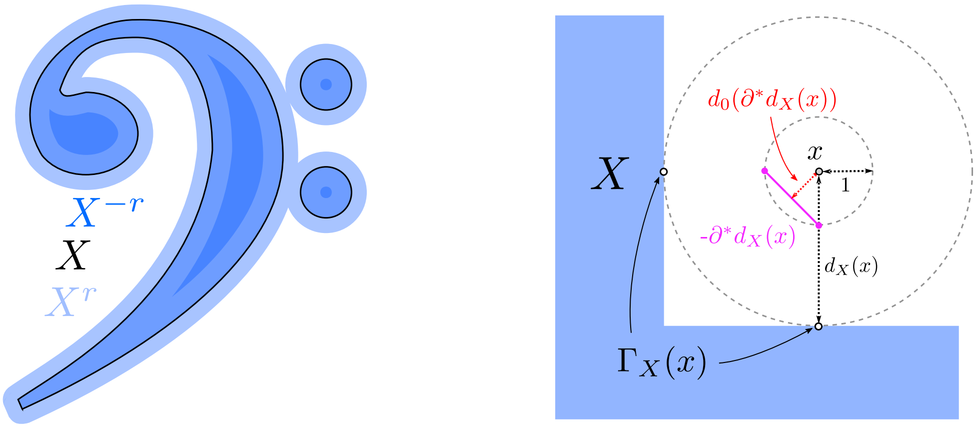
$$d_0(X) := \inf\{||x|| \mid x \in X\}\,.$$

  It measures how far X is from intersecting $\{0\}$.

- Given a function $f : \mathbb{R}^d \to \mathbb{R}$ locally Lipschitz, define $\partial^* f(x)$ its *Clarke Gradient at x* as the convex hull of limits of the form $\lim_{h\to 0} \nabla f(x+h)$ - see Section 2.4. In particular, if $f = d_X$ and if x lies outside of X, $-\partial^* d_X(x)$ is the convex hull of the directions to the points $z \in X$ such that $d_X(x) = ||x-z||$.

$$\partial^* d_X(x) := \mathrm{Conv}\left(\left\{\frac{x-z}{||x-z||} \mid z \in \Gamma_X(x)\right\}\right)$$

  where such z form the set $\Gamma_X(x)$ of *closest points* to x in X (cf Figure 1, right). Elements of $\Gamma_X(x)$ will be denoted by the letter ξ. In particular, we denote $\xi_X(x)$ *the* closest point to x in X when $\Gamma_X(x)$ is a singleton.



A bass clef X inflated (X^r) and eroded (X^{-r}) Clarke gradient of d_X outside of X

FIGURE 1: Offsets of X and Clarke gradient of d_X outside of X.

- Given μ in $(0, 1]$, define the *μ-reach* of a subset X of $\mathbb{R}^d$:

$$\operatorname{reach}_\mu(X) \coloneqq \sup\left(\left\{s \in \mathbb{R} \mid d_X(x) \leq s \implies d_0(\partial^* d_X(x)) \geq \mu\right\}\right) \tag{2.1}$$

  Having $\operatorname{reach}_\mu(X) > 0$ means that in a certain neighborhood of X, the angles between two closest point in X cannot be too flat. The lower the μ, the flatter allowed. Note that this definition coincides with the classical one found in geometric inference as $d_0(\partial^* d_X(x))$ is exactly the norm of the generalized gradient $\nabla d_X(x)$ defined by Lieutier in [4].

  Throughout this article, when no value of μ has been fixed, for any closed $X \subset \mathbb{R}^d$, *having a positive μ-reach* means that there is a certain $\mu \in (0, 1]$ with $\operatorname{reach}_\mu(X) > 0$. This class of sets is certainly broad, intuitively containing stratified sets whose corners are not infinitely pointy. A corollary from Lemma 1.6 in Fu [5] is that for any subanalytic set $X \subset \mathbb{R}^d$, the set of value $r > 0$ such that X^r has not a positive μ-reach is finite.

- The *reach* of a subset of $\mathbb{R}^d$ is a quantity first studied by Federer in [6] coinciding with reach_1. It is the largest number t such that $d_X(x) < t$ implies that x has a unique closest point in X. The class of sets with positive reach have been studied for a long time - see [?] for a broad overview. This class notably contains convex sets and submanifolds of Euclidean spaces.

  When X has a positive reach_μ the complement sets of small offsets of X have positive reach.

Theorem 2.1: Reach of complement of offsets (Chazal et al. [7], 4.1)

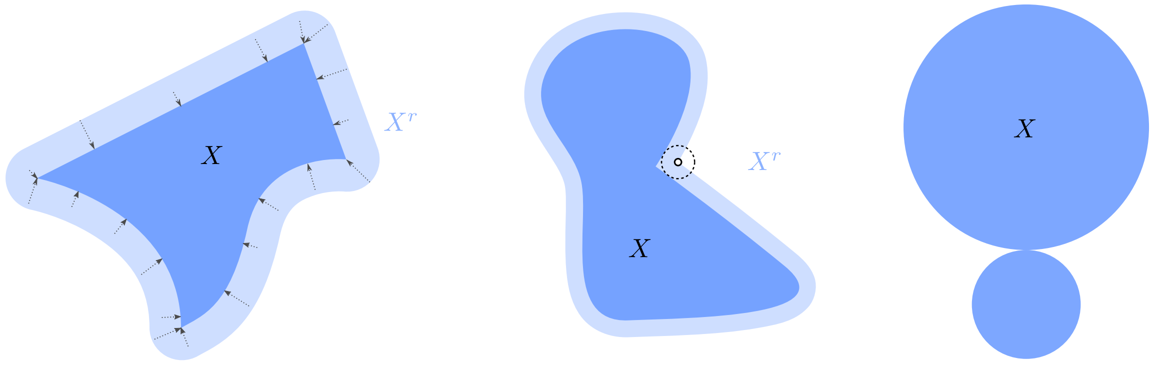
Let X be compact subset of $\mathbb{R}^d$, $\mu \in (0, 1]$ and $0 < r < \operatorname{reach}_\mu(X)$.

Then $\operatorname{reach}(^\neg(X^r)) \geq \mu r$.

- The *Tangent Cone* of X at x, $\operatorname{Tan}(X, x)$ is defined as the cone generated by the limits $\lim_{n\to\infty} \frac{x_n - x}{||x_n - x||}$, where the sequence $(x_n)_{n\in\mathbb{N}}$ belongs in X, tends to x and never takes the value x. In that case, we say that u is *represented* by the sequence $(x_n)_{n\in\mathbb{N}}$.

  When $X \subset \mathbb{R}^d$ has positive reach, $\operatorname{Tan}(X, x)$ is a convex cone which can be characterized as follows for any $x \in X$:

$$\operatorname{Tan}(X, x) = \left\{u \in \mathbb{R}^d \;\middle|\; \lim_{t\to 0^+} \frac{d_X(x + tu)}{t} = 0\right\}$$



reach$(X) > r > 0$ reach$(X) = 0$ reach$_\mu(X) = 0$ for all $\mu \in (0, 1]$.
reach$_\mu > 0$ for some $1 > \mu > 0$.

FIGURE 2: Sets with particular reach$_\mu$.

- When X has positive reach, define $\mathrm{Nor}(X, x)$ its *normal cone* at x as the set dual to the tangent cone at x:
$$\mathrm{Nor}(X, x) \coloneqq \mathrm{Tan}(X, x)^{\circ}.$$
It is related to the projection to the closest point in X function ξ_X by the following characterisation, for any $0 < t < \mathrm{reach}(X)$:
$$\mathrm{Nor}(X, x) \cap \mathbb{S}^{d-1} = \left\{u \in \mathbb{S}^{d-1} \mid \xi_X(x + tu) = x\right\}$$

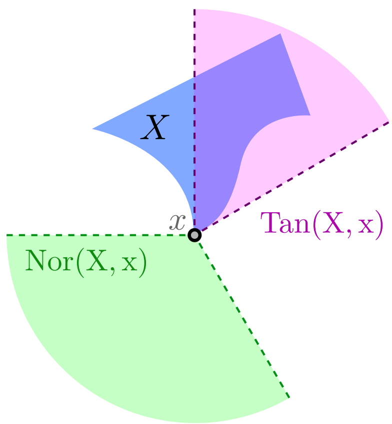


FIGURE 3: Tangent and Normal cones of X at x when reach$(X) > 0$

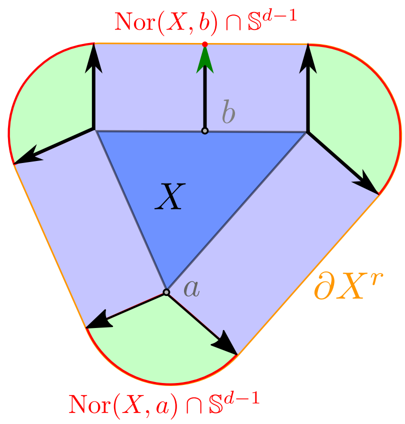


FIGURE 4: Some unit normal cones (in red) when $0 < r < \mathrm{reach}(X)$

- If $X \subset \mathbb{R}^d$ has positive reach, we say that X is *full dimensional* when every $\mathrm{Tan}(X, x)$ has dimension d for every $x \in \partial X$, which is characterized by the following condition on the normal cones:
$$(x, n) \in \partial X \times \mathrm{Nor}(X, x) \implies -n \notin \mathrm{Nor}(X, x)$$

2.2 Normal bundles

We are now in position to define the *normal bundle* of sets with positive reach or whose complement sets have positive reach.

Definition 2.2: Normal cones and normal bundles

- *When* $\neg X$ *has positive reach, define its normal cone at* x *via:*

$$\mathrm{Nor}(X, x) := -\mathrm{Nor}(\neg X, x)$$

This definition is consistent when both $\neg X$ *and* X *have positive reach.*

- *In case* X *or* $\neg X$ *has positive reach, its* unit normal bundle *is defined as follows:*

$$\mathrm{Nor}(X) := \bigcup_{x \in \partial X} \{x\} \times (\mathrm{Nor}(X, x) \cap \mathbb{S}^{d-1})$$

- *A pair* $(x, n) \in \mathrm{Nor}(X)$ *is said to be* regular *when* $\mathrm{Tan}(\mathrm{Nor}(X), (x, n))$ *is a* $(d-1)$ *dimensional vector space.*

Proposition 2.3: Almost all pairs of $\mathrm{Nor}(X)$ are regular

When either X *or* $\neg X$ *has positive reach,*

- $\mathrm{Nor}(X)$ *is a* $(d-1)$*-lipschitz submanifold of* $\mathbb{R}^d \times \mathbb{S}^{d-1}$*;*
- *Pairs* $(x, n) \in \mathrm{Nor}(X)$ *are regular* $\mathcal{H}^{d-1}$*-almost everywhere, where* $\mathcal{H}^{d-1}$ *is the* $(d-1)$*-Hausdorff measure on* $\mathbb{R}^d \times \mathbb{S}^{d-1}$*.*

Proof. Assume $\mathrm{reach}(X) > 0$ and let $0 < r < \mathrm{reach}(X)$. The map $\mathrm{Nor}(X) \to \partial X^r, (x, n) \mapsto (x + rn)$ is bilipschitz and ∂X is a $C^1(d-1)$ submanifold of $\mathbb{R}^d$ by the implicit function theorem. Else, let $0 < r < \mathrm{reach}(\neg X)$. The map $\mathrm{Nor}(X) \to \partial X^{-r}, (x, n) \mapsto (x + rn)$ is bilipschitz and the same reasoning stands. □

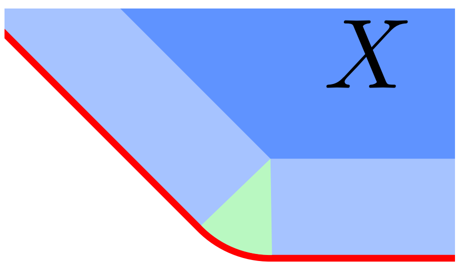


FIGURE 5: Normal Bundle of X with $\mathrm{reach}(X) > 0$

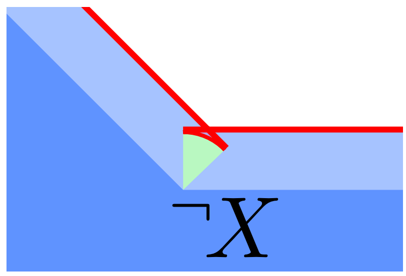


FIGURE 6: Normal Bundle of $\neg X$ with $\mathrm{reach}(X) > 0$

The construction of $\mathrm{Nor}(X)$ stems from the more general concept of *normal cycle* of a set [8, 5]. While we do not need to write our hypothesis using this more involved language, in our case the normal bundle is the support of a $(d-1)$ Legendrian cycle over $\mathbb{R}^d \times \mathbb{S}^{d-1}$, whose tangent spaces' structure is already known.

Proposition 2.4: Tangent spaces of Normal Bundles (Rataj & Zähle, 2019 [9])

Let X *be a compact set admitting a normal bundle* $\mathrm{Nor}(X)$*.*
Then for any regular pair $(x, n) \in \mathrm{Nor}(X)$*, there exist*

- *A family* $\kappa_1, \ldots, \kappa_{d-1}$ *in* $\mathbb{R} \cup \{\infty\}$ *called* principal curvatures at (x, n)
- *A family* $b_1, \ldots, b_{d-1} \in \mathbb{R}^d$ *of vectors orthogonal to* n *called* principal directions *at*

(x,n) *such that the family* $\left(\frac{1}{\sqrt{1+\kappa_i}^2}b_i, \frac{\kappa_i}{\sqrt{1+\kappa_i^2}}b_i\right)_{1\le i\le d-1}$ *form an orthonormal basis of* $\mathrm{Tan}(\mathrm{Nor}(X),(x,n))$.

Moreover,

- *Principal curvatures are unique up to permutations.*
- *Principal directions* b_i *associated to* κ_i *are unique up to the determination of an orthonormal basis of* $\ker(u,v\mapsto u-\kappa_i v)$ *if* $\kappa_i<\infty$*, or* $\ker(u,v\mapsto v)$ *if* $\kappa_i=\infty$.

These principal curvatures coincide with the ones found in differential geometry as eigenvalues of the second fundamental form. Indeed, assume that $X\subset\mathbb{R}^d$ is bounded by a $C^{1,1}$-hypersurface, i.e the boundary of X is an hypersurface such that the Gauss map $x\in\partial X\mapsto n(x)\in\mathbb{S}^{d-1}$ is Lipschitz. The pair $(x,n(x))\in\mathrm{Nor}(X)$ is regular if and only if n is differentiable at x [3]. In that case, its differential is symmetric and its eigenvalues counted with multiplicity (resp. orthonormal basis of eigenvectors) are principal curvatures (resp. principal directions) at $(x,n(x))$.

2.3 Critical points and Hessians for $f_{|X}$

In the paper *Curvature measures and Generalized Morse Theory* [3], Fu defines a notion of Morse Functions over sets of positive reach and prove the Morse theorems for them. The part of this paper focusing on generalized Morse theory forms a basis of our reasoning in Section 3. We will use the same definitions of critical points and hessians, which we now recall. The projection $\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ onto the first factor is denoted π_0.

Definition 2.5: Critical points and Hessian

Let $f:\mathbb{R}^d\to\mathbb{R}$ *be smooth and* X *be a set of* $\mathbb{R}^d$ *admitting a normal bundle.*

- *Let* $(x,n)\in\partial X\times\mathbb{S}^{d-1}$ *be regular as in Proposition 2.4. Take* (b_i) *an orthonormal basis of* $\pi_0(\mathrm{Tan}(\mathrm{Nor}(X),X,(x,n)))$ *consisting of all principal directions with finite associated curvatures. The second fundamental form* $\mathrm{I\!I}_{x,n}$ *is defined as the bilinear form on* $\pi_0(\mathrm{Tan}(\mathrm{Nor}(X),(x,n)))$ *such that:*

$$\mathrm{I\!I}_{x,n}(b_i,b_j) \coloneqq \kappa_i\delta_{i,j} \tag{2.2}$$

  which generalizes the classical fundamental form obtained when X *has a smooth boundary.*

- $x\in X$ *is a* critical point *of* $f_{|X}$ *when* $\nabla f(x)\in\mathrm{Nor}(X,x)$
- $c\in\mathbb{R}$ *is a* critical value *of* $f_{|X}$ *when* $f^{-1}(c)$ *contains at least a critical point of* $f_{|X}$*. Otherwise,* c *is a* regular value *of* $f_{|X}$.
- *If* x *is a critical point of* $f_{|X}$ *with* $\nabla f(x)\neq 0$*, put* $n=\frac{-\nabla f(x)}{||\nabla f(x)||}$.

  If (x,n) *is regular,* the Hessian of $f_{|X}$ at x *is defined as a bilinear form over* $\pi_0(\mathrm{Tan}(N_X,(x,n)))$*:*

$$Hf_{|X}(x)(u,v)\coloneqq Hf(x)(u,v)+||\nabla f(x)||\,\mathrm{I\!I}_{x,n}(u,v)$$

- *The* index *of this Hessian is the dimension of the largest subspace on which* $Hf_{|X}$ *is negative definite.*

- *We say that a critical point x of $f_{|X}$ is* non-degenerate *when* $\nabla f(x) \neq 0$, (x, n) *is a regular pair of* $\mathrm{Nor}(X)$ *and its Hessian* $Hf_{|X}(x)$ *is not degenerate.*

- $f_{|X}$ *is said to be* Morse *when its critical points are non-degenerate.*

Using these definitions, Fu proved the Morse Theorems for sets with positive reach.

Theorem 2.6: *Generalized Morse Theory for sets with positive reach* (Fu, 1989)

Let X be a compact subset of $\mathbb{R}^d$ with positive reach and let $f : \mathbb{R}^d \to \mathbb{R}$ be a smooth function such that $f_{|X}$ is Morse with at most one critical value per level set.

Then for any regular value $c \in \mathbb{R}$, X_c has the homotopy type of a CW-Complex with one λ_p cell for each critical point p such that $f(p) < c$, where

$$\lambda_p = \textit{Index of } Hf_{|X} \textit{ at } p$$

2.4 Clarke gradients and approximate flows

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function. It is differentiable almost everywhere thanks to Rademacher's Theorem. Consider $\partial^* \phi(x)$ its *Clarke gradient* at x. It is a subset of $\mathbb{R}^d$ generalizing the gradient of ϕ defined as the convex hull of limits of the form $\nabla\phi(x+h), h \to 0$. A key property of Clarke Gradients is its upper semicontinuity, leading to the following proposition.

Proposition 2.7: Semicontinuity Clarke Gradients

Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function.
If a sequence $(x_i)_{i\in\mathbb{N}}$ converges to x, we have

$$\liminf_{i\to\infty} d_0\left(\partial^* \phi(x_i)\right) \geq d_0\left(\partial^* \phi(x)\right)$$

Assuming $\partial^* \phi(x)$ stays uniformly away from 0, we are able to build deformation retractions between the sublevel sets of ϕ using approximations of what would be the flow of $-\phi$ had it been smooth.

Proposition 2.8: Approximate flow of a Lipschitz function

Let $a < b \in \mathbb{R}$. Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a locally Lipschitz function lipschitz on $\overline{\phi^{-1}(a, b]}$. Assume that

$$\inf \{d_0(\partial^* \phi(x)), x \in \phi^{-1}(a, b]\} = \mu > 0$$

. Then for every $\varepsilon > 0$, there exists a continuous function

$$C_\phi : \begin{cases} [0,1] \times \phi^{-1}(]\infty, b]) & \to & \phi^{-1}(]-\infty, b]) \\ (t, x) & \mapsto & C_\phi(t, x) \end{cases}$$

such that

- *For any $s > t$ and x such that $C(s, x) \in \phi^{-1}(a, b]$, we have*

$$\phi(C_\phi(s, x)) - \phi(C_\phi(t, x)) \leq -(s - t)(b - a)$$

- *For any $t \in [0,1]$, $x \in \phi^{-1}(\infty, a]$, $C_\phi(t,x) = x$*
- *For any $x \in \phi^{-1}(-\infty, b]$, the map $s \mapsto C_\phi(s,x)$ is $\frac{b-a}{\mu-\varepsilon}$-lipschitz.*

In particular, $C_\phi(1,\cdot)$ is a deformation retraction between $\phi^{-1}(-\infty,a]$ and $\phi^{-1}(-\infty,b]$.

Proof. This proposition is the main statement in section D of [10] where the constants have been optimized and generalized to lipschitz functions instead of distance functions. For the sake of completeness, we write the full proof. Let $\varepsilon > 0$. Let $x \in \phi^{-1}(a,b]$ and take by semicontinuity of the Clarke gradients B_x an open ball centered in x such that $\partial^*\phi(y) \subset \partial^*\phi(x)^\varepsilon$ for any $y \in B_x$. Since $\partial^*\phi(x)$ is a closed convex set, put $W(x)$ the vector in $\partial^*\phi(x)$ realising $||W(x)|| = d_0(\partial^*\phi(x))$ - that is, the closest point to 0 in $\partial^*\phi(x)$. By convexity,

$$\forall u \in \partial^*\phi(x), \langle u, W(x)\rangle \geq ||W(x)||^2 . \tag{2.3}$$

The family $\{B_x\}_{x\in\phi^{-1}(a,b]}$ is an open covering of $\phi^{-1}(a,b]$. Thanks to paracompactness, there exists a locally finite partition of unity $(\rho_i)_{i\in I}$ subordinate to this family. The support of each ρ_i has to be included in some $B(x_i)$, where $x_i \in \phi^{-1}(a,b]$.

Define the vector field V as a smooth interpolation of normalized $-W$:

$$V(y) \coloneqq -\sum_{i\in I}\rho_i(y)\frac{W(x_i)}{||W(x_i)||} \tag{2.4}$$

Obviously $||V(x)|| \leq 1$. Now by classical results write C the flow of V defined on a maximal open domain $\mathbb{D}$ in $\phi^{-1}(a,b] \times \mathbb{R}^+$. For any $x \in \phi^{-1}(a,b]$ and any $\zeta \in \partial^*\phi(x)$, we have:

$$\langle \dot{C}_x(0), \zeta\rangle \leq -\sum_i \rho_i(||W(x_i)|| - \varepsilon) \leq -\mu + \varepsilon \tag{2.5}$$

Define $\mathbb{D}_x$ by $(\{x\}\times\mathbb{R}^+) \cap \mathbb{D} = \{x\} \times \mathbb{D}_x$ the maximal subset of $\mathbb{R}$ for which the flow starting at x is defined. The set $\mathbb{D}_x$ is connected in $\mathbb{R}^+$ and we put $s_x = \sup \mathbb{D}_x$. Now the trajectory $\mathbb{C}(\cdot,x)$ is 1-lipschitz, meaning the curve $s \mapsto C(s,x)$ is rectifiable. We can thus define $C(s_x,x) = \lim_{s\to s_x^-} C(s,x)$ as the endpoint of this curve. The function $\phi(C(\cdot,x)) : \overline{\mathbb{D}_x} \to [a,b]$ is lipschitz and thus differentiable almost everywhere. Without loss of generality we can assume that it is differentiable at 0. Since $C(\cdot,x)$ has non-vanishing gradient $V(x)$ at 0, ϕ admits a directional derivative $\phi'(x,V(x))$ in direction $V(x)$. Now the work of Clarke [11] states that when the directional derivative exists, the Clarke gradients acts like a maxing support set, that is:

$$\phi'(x,V(x)) \leq \max\left\{\langle\zeta, V(x)\rangle \;\middle|\; \zeta \in \partial^*\phi(x)\right\} \leq -\mu + \varepsilon \tag{2.6}$$

Any lipschitz function is absolutely continuous, thus when $s,t \in \mathbb{D}_x$ and $t \leq s$, integrating the previous inequality, we obtain:

$$\phi(C(s,x)) - \phi(C(t,x)) \leq -(\mu-\varepsilon)(s-t) \tag{2.7}$$

This yields $\phi(C(s_x,x)) = a$ and $s_x \leq \frac{b-a}{\mu-\varepsilon}$ for all $x \in \phi^{-1}(a,b]$. We extend the flow to $\phi^{-1}(-\infty,b]\times\mathbb{R}^+$ by putting

$$C(t,x) \coloneqq \begin{cases} C(\min(t,s_x),x) & \text{when } a < \phi(x) \leq b, \\ x & \text{else.}\end{cases}$$

It remains to show that C is continuous at every point $(x,s) \in \phi^{-1}(-\infty,b]\times\mathbb{R}^+$. Let K be a Lipschitz constant for ϕ over $\overline{\phi^{-1}(a,b]}$. Assume $s \geq s_x$. Let $c > 0$. For every $\delta > 0$, there exists $\rho_x(\delta) > 0$ such that for all $(y,t) \in B(x,\rho_x(\delta)) \times [0, s_x - x]$, we have both $(y,t) \in \mathbb{D}$ and $|\phi(y,t) - \phi(x,t)| \leq \delta$. Notably this implies $s_y > s_x - c$ and $\phi(C(y,s_x-c)) \leq a + \delta + kc$, which yields $s_y \leq s_x - c + \frac{kc+\delta}{\mu-\varepsilon}$. And finally, for any (y,t) such that $|s-t| \leq c$ and $||y-x|| \leq \rho_x(\delta)$, we have:

$$||C(y,t) - C(x,s)|| \leq$$
$$||C(y, \min(t, s_y)) - C(x, s_x - c)|| + ||C(y, s_x - c) - C(x, s_x - x)|| + ||C(x, s_x, -c) - C(x, s_x)||$$
$$\leq \frac{\delta+kc}{\mu-\varepsilon} + \delta + c$$

Now if $\phi(x) < a$, C is locally constant around (x,s) for any $s \geq 0$. Finally, if $\phi(x) = a$, the function $z \mapsto \max(a, \phi(z))$ is K-lipschitz and we thus have $\phi(y) > a \implies |\phi(x) - \phi(y)| \leq k\,||x-y||$ which means that $s_y \leq \frac{k||x-y||}{\mu-\varepsilon}$ and finally

$$||C(y,s) - C(x,s)|| \leq ||C(y,s) - y|| + ||y-x|| \leq \left(\frac{k}{\mu-\varepsilon} + 1\right)||x-y||$$

Finally we reparametrize C to obtain $C_\phi(t,x) = C\left(\frac{(b-a)t}{\mu-\varepsilon}, x\right)$ which yields an homotopy such that $\phi^{-1}(-\infty, a]$ is a strong deformation retraction of $\phi^{-1}(-\infty, b]$. □

2.5 Relating the Normal Cones to Clarke Gradients of distance functions

The normal bundle of X is related to d_X in the following fashion.

Theorem 2.9: Normal cones and the Clarke gradient of the distance function

Let $X \subset \mathbb{R}^d$ be such that $\text{reach}(^\neg X) > 0$ *and full dimensional. Let $x \in \partial X$.*
Then the normal cone of X at x is determined by the Clarke gradient of d_X at x:

$$\text{Nor}(X,x) = \text{Cone}\,\partial^* d_X(x)$$

Proof. Let $\text{reach}(^\neg X) > r > 0$. First remark that

$$\begin{aligned}
\partial^* d_{X^{-r}}(x) &= -\,\text{Conv}\left\{\frac{x-z}{||x-z||}, z \in X^{-r} \text{ with } d_X^{-r}(x) = ||z-x||\right\} \\
&= -\,\text{Conv}\{u \in \mathbb{S}^{d-1}, d_{^\neg X}(x+ru) = r\}) \\
&= -\,\text{Conv}\left(\text{Nor}(^\neg X, x) \cap \mathbb{S}^{d-1}\right)
\end{aligned}$$

On the other hand by definition, the Clarke gradient of $d_{X^{-r}}$ at x is determined locally by the gradients around x in every direction:

$$\partial^* d_{X^{-r}}(x) = \text{Conv}\left\{\lim \nabla d_{X^{-r}}(x_i) \mid (x_i) \in (\mathbb{R}^d)^{\mathbb{N}} \text{ converging to } x\right\}$$

Now compare to the Clarke Gradient of d_X for which the gradient contributing only come from directions outside of X (cf. [11], 2.5):

$$\partial^* d_X(x) = \text{Conv}\{0, \lim \nabla d_X(x_i) \mid (x_i) \in (\mathbb{R}^d)^{\mathbb{N}} \text{ converging to } x \text{ such that for all } i, d_X(x_i) > 0\,\}$$

Note that in both definition we implictly require x_i to be points where d_X is differentiable. On those points the gradients of d_X and $d_{X^{-r}}$ coincide, yielding

$$\text{Cone}\,\partial^* d_X(x) \subset -\,\text{Nor}(^\neg X, x). \tag{2.8}$$

The other inclusion $-\,\text{Nor}(^\neg X, x) \subset \text{Cone}\,\partial^* d_X(x)$ is Lemma 2.13 whose proof will be the remainder of this subsection. We will prove the opposite inclusion on their polar cones, that is

$$\partial^* d_X(x)^\circ \subset -\,\text{Nor}(^\neg X, x)^\circ = -\,\text{Tan}(^\neg X, x). \tag{2.9}$$

□

Lemma 2.10: Tangent cone stability under addition with $\partial^* d_X(x)$

Let $X \subset \mathbb{R}^d$ and $x \in \partial X$. Let $u \in \partial^* d_X(x)^o$, Then for all $h \in \mathrm{Tan}(X,x)$, $u+h \in \mathrm{Tan}(X,x)$.

Proof. We use Clarke's [11] characterization of the dual cone to the Clarke gradient:

$$\partial^* d_X(x)^{\mathrm{o}} = \left\{ u \;\middle|\; \lim_{\substack{x_h \to x \\ x_h \in X}} \lim_{\delta \to 0^+} \frac{1}{\delta} d_X(x_h + \delta u) = 0 \right\} \tag{2.10}$$

Consider the following modulus of continuity:

$$\omega_u(\varepsilon, \lambda) := \sup_{\substack{x_h \in X \\ ||x - x_h|| \le \varepsilon}} \sup_{0 < \delta \le \lambda} \frac{d_X(x_h + \delta u)}{\delta}$$

When u belongs $\partial^* d_X(x)^{\mathrm{o}}$, Clarke's characterization 2.10 implies that $\omega_u(\varepsilon, \delta) \to 0$ as $\varepsilon, \delta \to 0$.

Now take a sequence $x_i \to x$ representing any $h \in \mathrm{Tan}(\partial X, x)$. Put $\varepsilon_i = ||x - x_i||$ and consider the sequence $x_i + \varepsilon_i u$. Take ξ_i in $\Gamma_X(x_i + \varepsilon_i u)$:

$$||\xi_i - x_i - \varepsilon_i u|| = d_X(x_i + \varepsilon_i u) \le \varepsilon_i \omega(\varepsilon_i, \varepsilon_i)$$

Thus we can write

$$\xi_i - x = \varepsilon_i(h + o(1) + u + O(\omega(\varepsilon_i, \varepsilon_i))) = \varepsilon_i(u + h + o(1))$$

showing that ξ_i is a sequence in X representing $u + h$. □

Lemma 2.11: Intersection of complement tangent spaces

Let $X \subset \mathbb{R}^d$. Then

$$\mathrm{Tan}(\partial X, x) = \mathrm{Tan}(X, x) \cap \mathrm{Tan}(\neg X, x)$$

Proof. We have to prove that $\mathrm{Tan}(X, x) \cap \mathrm{Tan}(\neg X, x)$ is included in $\mathrm{Tan}(\partial X, x)$.

Let $u \in \mathrm{Tan}(X, x) \cap \mathrm{Tan}(\neg X, x)$ be of norm 1. Take a sequence x_n (resp. $\neg x_n$) in X (resp. $\neg X$) representing u, i.e such that

$$\begin{aligned} x_n &= x + ||x_n - x||\,(u + o(1)\,) \\ \neg x_n &= x + ||\neg x_n - x||\,(u + o(1)\,). \end{aligned}$$

The segment $[x_n, \neg x_n]$ has to intersect ∂X, which means that there exists a $\lambda_n \in [0,1]$ such that $\partial x_n = \lambda_n x_n + (1 - \lambda_n) \neg x_n$ belongs in ∂X. This yields

$$\begin{aligned} \partial x_n - x &= (\lambda_n\,||x_n - x|| + (1 - \lambda_n)\,||\neg x_n - x||\,)\,(u + o(1)\,) \\ &= ||\partial x_n - x||\,(u + o(1)\,) \end{aligned}$$

meaning that u is represented by a sequence in ∂X. □

Lemma 2.12: Complement tangent cone are tangent cone of complement

Let $X \subset \mathbb{R}^d$ be a closed set such that either X or $\neg X$ has positive reach and let $x \in \partial X$. We have

$$\neg\,\mathrm{Tan}(\neg X, x) = \mathrm{Tan}(X, x)$$

Proof. Without loss of generality, assume $\mathrm{reach}(^\neg X) > 0$. Since $\mathrm{Tan}(X,x) \cup \mathrm{Tan}(^\neg X, x) = \mathbb{R}^d$, we know that $^\neg\,\mathrm{Tan}(^\neg X, x) \subset \mathrm{Tan}(X,x)$. We will show the opposite inclusion by proving that $\mathrm{Tan}(X,x) \cap \mathrm{int}(\mathrm{Tan}(^\neg X, x)) = \emptyset$.

Let $u \in \mathrm{Tan}(X,x) \cap \mathrm{int}(\mathrm{Tan}(^\neg X, x))$. Then it belongs in $\mathrm{Tan}(\partial X, x)$ by Lemma 2.11. Take a sequence $x_n \in \partial X$ such that $\frac{x_n - x}{||x_n - x||} \to u$. Take a sequence $v_n \in \mathrm{Nor}(^\neg X, x_n)$. Fix a $\lambda \in (0, \mathrm{reach}(^\neg X))$. We have

$$\mathrm{int}(B(x_n + \lambda v_n, \lambda)) \cap ^\neg X = \emptyset \tag{2.11}$$

Since $u \in \mathrm{int}(\mathrm{Tan}(^\neg X, x))$, there exists a $\lambda' \in (0, \lambda)$ such that for any n large enough

$$\frac{x_n - x}{||x_n - x||} + \lambda' v_n \in \mathrm{Tan}(^\neg X, x)$$

Consider for any such n a sequence $(y_{m,n})_{m\in\mathbb{N}} \in ^\neg X$ representing the previous vector. We will now prove that $y_{m,n}$ cannot be in $^\neg X$ for large m, n as it represents a infinitesimal version of vector of ∂X shifted in a direction normal to X. We can write

$$y_{m,n} = x + ||y_{m,n} - x|| \left(\frac{x_n - x}{||x_n - x||} + \lambda' v_n + \omega_{m,n} \right)$$

with $\omega_{m,n} \to_{m\to\infty} 0$ for every n.

$$\begin{aligned} ||y_{m,n} - x_n - \lambda' v_n|| = & \left|\left| \left(\lambda - \lambda')v_n + (x_n - x)(1 - \frac{||y_{m,n} - x||}{||x_n - x||} \right) + ||y_{m,n} - x||\, \omega_{m,n} \right|\right| \\ \leq & (\lambda - \lambda') + ||x_n - x|| + ||y_{m,n} - x||\, (\omega_{m,n} - 1) \end{aligned}$$

The last quantity is strictly smaller than λ for m, n large enough, contradicting 2.11. ☐

Lemma 2.13: Relationship between normal cones and Clarke Gradients

Let $X \subset \mathbb{R}^d$ such that $\mathrm{reach}(^\neg X) > 0$. Then if $\mathrm{Tan}(^\neg X, x)$ has full dimension, we have:

$$\partial^* d_X(x)^o \subset -\,\mathrm{Tan}(^\neg X, x)$$

In particular, this full-dimensional condition is verified for all $x \in \partial X$ when X is a Lipschitz submanifold.

Proof. Let $u \in \partial^* d_X(x)^o$. By Lemma 2.10 we know that

$$u + \mathrm{Tan}(X,x) \subset \mathrm{Tan}(X,x)$$

which amounts to

$$\mathbb{R}^d \setminus (u + \mathrm{Tan}(X,x)) \supset \mathbb{R}^d \setminus \mathrm{Tan}(X,x)$$

From Lemma 2.12, we know that $^\neg\,\mathrm{Tan}(X,x) = \mathrm{Tan}(^\neg X, x)$ by the full dimensionality condition. This yields

$$u + \mathrm{int}(\mathrm{Tan}(^\neg X, x)) \supset \mathrm{int}(\mathrm{Tan}(^\neg X, x))$$

now taking the closure of both sides, along with the full-dimensionality condition, ensures the inclusion

$$u + \mathrm{Tan}(^\neg X, x) \supset \mathrm{Tan}(^\neg X, x)$$

which implies that u belongs in $-\,\mathrm{Tan}(^\neg X)$. ☐

3 Morse Theory for completentary regular sets

In this section, we use the previous tools and propositions to infer the two Morse theorems when X is *complementary regular* (cf. Section 3.1) and f is Morse (in the sense of Definition 2.5). In this setting, the eroded sets X^{-r} converge to X in the Hausdorff sense when r tends to 0 and they are $C^{1,1}$ by the implicit function theorem when $r < \text{reach}(\neg X)$.

Our approach is as follows. Let c be a regular value of $f_{|X}$. Consider a family of functions $f_{r,c}$ converging to f, in a way we will later precise, as r tends to 0. When $r = 0$, our notations are consistent with $f_{0,c} = f$. Consider the sublevel sets:

$$X_c = X \cap f^{-1}(-\infty, c] \quad \text{and} \quad X_c^{-r} := X^{-r} \cap f_{r,c}^{-1}(-\infty, c]$$

and remark that they are the zero sublevel sets of the following functions:

$$\phi_r = d_{X^{-r}} + \max(f_{r,c} - c, 0).$$

- In Section 3.1, we define the regularity condition required on sets $X \subset \mathbb{R}^d$ to prove the Morse Theorems. Such sets are called *complementary regular*. Proposition 3.2 describes how most offsets are part of this class.
- In Section 3.2, we prove that there exists a $K > 0$ such that there exists a retraction of any tubular neighborhoods $(X_c^{-r})^K$ onto X when $r > 0$ is small enough. We prove a technical lemma to ensure that we can build an approximate inverse flow of $\phi_{r,c}$ using Proposition 2.8.
- In Section 3.3 we study the case $r = 0$ and prove that for $\varepsilon > 0$ small enough, the sets X_{c+a} can be retracted onto $X_{c-\varepsilon}$ also using Proposition 2.8.
- In Section 3.4 we let c be a critical value and assume there is only one corresponding critical point which is be non-degenerate. We show that for any $\varepsilon > 0$ the change in homology between $X_{c+\varepsilon}$ and $X_{c-\varepsilon}$ is determined by the curvature of X at the pair $\left(x, \frac{\nabla f(x)}{||\nabla f(x)||}\right)$ and the Hessian of $f_{|X}$ at x. We prove this by considering $f_{r,c}$ to be f translated with magnitude r in the direction $-\nabla f(x)$.
- In Section 3.5 we let c be a critical value and assume that the critical points in $f^{-1}(c)$ are non-degenerate although there might be several of them. We determine the topology changes between $X_{c-\varepsilon}$ and $X_{c+\varepsilon}$ through the curvature of X by considering a more involved $f_{r,c}$.

3.1 Complementary regular sets

We will prove the Morse theorems on *complementary regular sets*, which are defined as follows.

Definition 3.1: Complementary regular sets

We say that $X \subset \mathbb{R}^d$ is a complementary regular set *when it verifies the following four conditions:*

- $\text{reach}(\neg X) > 0$*;*
- $\exists\, \mu \in (0, 1]$ *such that* $\text{reach}_\mu(X) > 0$*;*
- *X is compact;*
- *X has full dimension.*

The following proposition gives mild sufficient conditions for offsets to be complementary regular.

Proposition 3.2: Offsets among complementary regular sets

1. *Let $\mu \in (0,1], Y \subset \mathbb{R}^d$ and $\varepsilon \in \mathbb{R}$ be such that* $\mathrm{reach}_\mu(Y) > \varepsilon \geq 0$. *Then $X = Y^\varepsilon$ is a complementary regular set.*

2. *Let Z be a compact subanalytic subset of $\mathbb{R}^d$. For all $\varepsilon > 0$ but a finite number, Z^ε is a complementary regular set.*

Proof. 1. Let Y be any set verifying the assumptions. From Theorem 4.1 in Chazal et al. [7], $\mathrm{reach}(\neg(Y^\varepsilon)) > 0$. Now Y^ε is a lipschitz domain thanks to Clarke's Inversion Theorem for Lipschitz functions, ensuring Y^ε has full dimension.

2. From Fu [5], we know that the set $\mathrm{Crit}(Z) = \{\varepsilon > 0 \mid \forall \mu \in (0,1],\ \mathrm{reach}_\mu(Z^\varepsilon) = 0\}$ is locally finite. If this set were to be unbounded, the diameter of Z would be infinite by the characterization of $\partial^* d_X$ in term of closest points, which contradicts the compacity assumption. By the previous result, for any $\varepsilon \in \mathrm{int}(\mathbb{R} \setminus \mathrm{Crit}(Z)) = \mathbb{R}^+ \setminus \mathrm{Crit}(Z)$, the offset Z^ε is a complementary regular set.

□

3.2 Building a deformation retraction between X_c and its smooth surrogate

Let $c \in \mathbb{R}$ be a regular value of $f_{|X}$. For any $r > 0$, we build a smoothed out version of X_c which we denote X_c^{-r}, close to X_c at rate $O(r)$ with respect to the Hausdorff distance. We define X_c^{-r} as the intersection between X^{-r} and the sublevel set of a family of functions $f_{r,c}$ defined as f translated with magnitude at most r. It will be denoted f_r to ease notation as c will be fixed. The direction of translation does not matter when c is a regular value, as shows the following lemma.

Lemma 3.3: Deformation retractions around X_c, X_c^{-r}

Let X be a complementary regular set. Let c be a regular value of $f_{|X}$ and $f_r = f \circ \gamma_r$ be f composed with a smooth function γ_r such that $\gamma_r(x) = x + r\eta(x)$ where $\eta, \nabla\eta$ are bounded on $\mathbb{R}^d$. Put $\phi = d_X + \max(f - c, 0)$ and $\phi_r = d_{X^{-r}} + \max(f_r - c, 0)$.

Then there exists $K > 0, M \geq 1, L \geq 1$ and piecewise-smooth flows

$$C : [0,1] \times \phi^{-1}(]-\infty, K]) \to \phi^{-1}(]-\infty, K])$$

$$C^r : [0,1] \times \phi_r^{-1}(]-\infty, K]) \to \phi_r^{-1}(]-\infty, K])$$

such that:

- *For all $r > 0$ small enough, $(X_c)^{\frac{K}{M}} \subset \phi_r^{-1}(-\infty, K]$ and $(X_c^{-r})^{\frac{K}{M}} \subset \phi^{-1}(-\infty, K]$*
- *$C(0,\cdot), C^r(0,\cdot)$ are identity over their respective spaces of definition;*
- *$C(1, (X_c)^{\frac{K}{M}}) = X_c$ and $C^r(1, (X_c^{-r})^{\frac{K}{M}}) = X_c^{-r}$*
- *For any $t \in [0,1]$, $C(t,\cdot)_{|X_c}$, $C^r(t,\cdot)_{|X_c^{-r}}$ are the identity over X_c and X_c^{-r}.*
- *$C(\cdot, x)$ and $C^r(\cdot, x)$ are $2KL$-Lipschitz in the first parameter when $r > 0$ is small enough, with $L = \sup\{d_0(\partial^*\phi(y))^{-1} \mid y \in \phi^{-1}(0, K]\}$*

Proof. Remark that $X_c = \phi_c^{-1}(0)$ and X_c^{-r} with $(\phi_c^r)^{-1}(0)$. We want to apply Proposition 2.8.

Define

$$\omega(s,K) := \inf_{\substack{r\in[0,s]\\ x\in\phi_r^{-1}(0,K]}} d_0(\partial^*\phi_r(x))$$

Now Lemma 3.5 states that

$$\liminf_{\substack{s\to 0^+\\ K\to 0^+}} \omega(s,K) > 0 \tag{3.1}$$

Take $K, s > 0$ small enough that for all $r \in [0,s]$, $\partial^*\phi_r$ does not vanish on $\phi_r^{-1}(0,K]$, allowing the offsets to be retracted by Proposition 2.8. The first derivatives of the flow are bounded by $l_{r,K} = \sup\{d_0^{-1}(\partial^*\phi_s(y)), s\in[0,r], y\in\phi_r^{-1}(0,K]\}$ which is finite when r, K are taken small enough and tend to L when r, K go to zero. Reparametrizing the flow as in the proof of Proposition 2.8, we can choose C, C^r to be $\frac{1+\varepsilon}{KL}$ Lipschitz for any $\varepsilon > 0$.

The functions $(\phi_r)_{r\in[0,s]}$ are uniformly Lipschitz. Consider $M = 1+\sup\{\mathrm{Lip}(\phi_r)_{r\in[0,s]}\}$. Since the sets X_c^{-t} converge to X_c in the Hausdorff sense when t goes to 0, and since $||\phi-\phi_r|| = O(r)$, we have

$$(X_c^{-t})^{\frac{K}{M}} \subset \phi_r^{-1}(0,K]$$

for any t, r small enough. □

Corollary 3.4: Homotopy Equivalence

Let c be a regular value of $f_{|X}$ such that $f_r = f\circ\gamma_r$ with $\gamma_r(x) = x + r\eta(x)$ with $\eta, \nabla\eta$ smooth and bounded on $\mathbb{R}^d$.
Then for all $r > 0$ small enough, X_c^{-r} and X_c have the same homotopy type.

Proof. Since $\lim_{r\to 0} d_H(X_c^{-r}, X_c) = 0$, the flows C, C_r are respectively well defined on X_c^{-r}, X_c for r small enough thanks to the previous lemma. Then $C(1,\cdot)\circ C^r(1,\cdot)$ (resp. $C^r(1,\cdot)\circ C(1,\cdot)$) is homotopic to Id_{X_c} (resp. $\mathrm{Id}_{X_c^{-r}}$) via the homotopy $(t,x)\mapsto C(1, C(t, C^r(t,x)))$ □

Lemma 3.5: Non vanishing $\partial^*\phi_r$ around a critical value

Let $r_i, K_i \to 0^+$, $x_i \in \phi_{r_i}^{-1}(0,K_i]$ and c be a regular value of $f_{[X}$. Then,

$$\liminf_{i\to\infty} d_0(\partial^*\phi_{r_i}(x_i)) > 0$$

Proof. We distinguish 7 cases to compute $\partial^*\phi_{r_i}(x_i)$. By extracting subsequences we can assume that (x_i) lies in one of this case. They are depicted in Figure 7.

In fact, we will show that for any such sequence, we have

$$\liminf_{i\to\infty} d_0(\partial^*\phi_{r_i}(x_i)) \geq \min(\mu,\sigma,\kappa) > 0 \tag{3.2}$$

where

- $\kappa := \inf_{f^{-1}(c)\cap X} ||\nabla f||$. It is a positive quantity because c is a regular value of $f_{|X}$.
- $\sigma := \inf_{x\in\partial X\cap f^{-1}(c)} d_0(A_x)$ where $x\mapsto A_x$ is the upper semi-continuous set-valued application defined by:

$$\begin{aligned}A_x &:= \left\{\lambda u + \nabla f(x) \mid \lambda\in[0,1], u\in\partial^* d_X(x)\right\} \cup \left\{u + \lambda f(x) \mid \lambda\in[0,1], u\in\partial^* d_X(x)\right\}\\ &= \Big([0,1]\cdot\partial^* d_X(x) + \{\nabla f(x)\}\Big) \cup \Big(\partial^* d_X(x) + [0,1]\cdot\{\nabla f(x)\}\Big)\end{aligned}$$

For any point $x \in \partial X$, keep in mind that from Theorem 2.9 we have the identity

$$\operatorname{Cone} \partial^* d_X(x) = \operatorname{Nor}(X, x)$$

which means that any direction in $\partial^* d_X(x)$ is a direction in $\operatorname{Nor}(X, x)$. The constant σ is positive because c is a regular value of $f_{|X}$, $\partial X \cap f^{-1}(c)$ is a compact set and the map $x \mapsto d_0(A_x)$ is lower semicontinuous. If it were to be zero, there would be a point $x \in \partial X \cap f^{-1}(c)$ with $d_0(A_x) = 0$. This would mean that the direction of $\nabla f(x)$ meets $\operatorname{Nor}(X, x)$, which contradicts the fact that c is a regular value.

- $\mu \leq \inf_{t \to 0}\{d_0(\partial^* d_X(x)) \mid 0 < d_X(x) < t\}$ is positive by hypothesis.

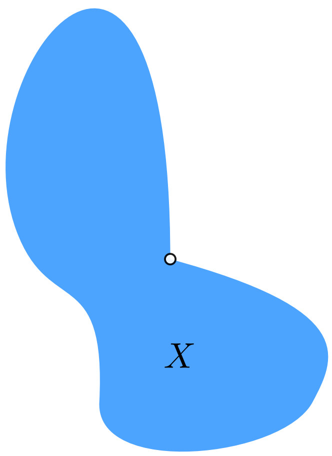
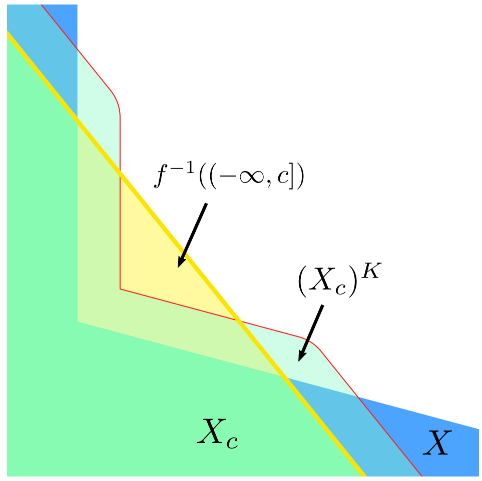


Illustration. X is a compact of $\mathbb{R}^2$ with $\operatorname{reach}_\mu(X) > 0$ for some $\mu > 0$.



Zoomed-in depiction of $X_c = X \cap f^{-1}(-\infty, c]$ and a tubular neighborhood $(X_c)^K$, $K > 0$ where f is a linear form.

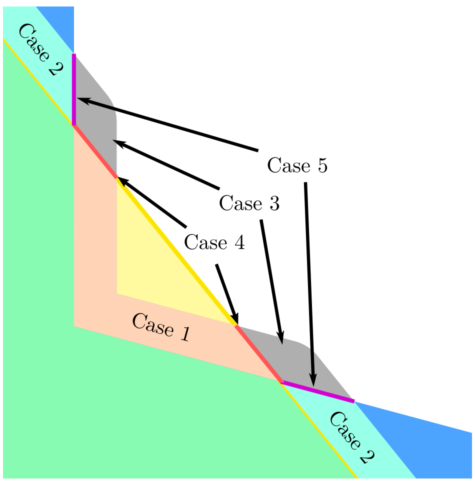
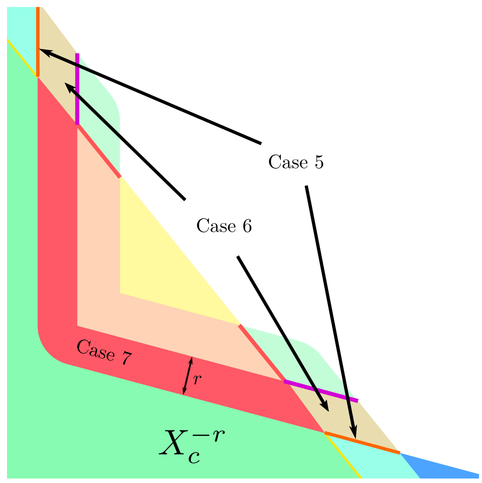


Illustration of cases 1 to 5 when $r = 0$. Cases 1 to 4 are defined independently of r.



Cases 5, 6 and 7 when $r > 0$.

FIGURE 7: Illustration of the 7 cases of Lemma 3.5.

Idea behind the proof. For each of the following cases, $\liminf_{i\to\infty} d_0(\partial^* \phi_{r_i}(x_i))$ is greater than one of the constants. The computation of $\partial^* \phi_{r_i}(x_i)$ shows that it either lies close to $\nabla f(x_i)$ or $\partial^* d_X(x_i)$ or close to be inside A_{x_i}. To ease some notations we write $\nu(x) = \frac{x}{||x||}$.

Case 1. $d_{X^{-r_i}}(x_i) > r_i$ and $f_{r_i}(x_i) < c$.

Then $\partial^* \phi_{r_i}(x_i) = \partial^* d_X(x_i)$ with $0 < d_X(x_i) < K_i$. By the μ-reach hypothesis, we have

$$\liminf_{i\to\infty} d_0(\partial^* \phi_{r_i}(x_i)) \geq \mu > 0. \tag{3.3}$$

Case 2. $x_i \in \text{int}(X^{-r_i})$.

Then $\partial^* \phi_{r_i}(x_i) = \nabla f_{r_i}(x_i)$ and $0 < f_{r_i}(x_i) - c \leq K_i$. Since $||\nabla f_{r_i}(x_i) - \nabla f(x_i)|| = O(r_i)$, we have

$$\liminf_{i\to\infty} d_0(\partial^* \phi_{r_i}(x_i)) \geq \kappa > 0. \tag{3.4}$$

Case 3. $d_{X^{-r_i}}(x_i) > r_i$ and $f_{r_i}(x_i) > c$.

Then $\partial^* \phi_{r_i}(x_i) = \partial^* d_X(x_i) + \nabla f(x_i) \subset A_{x_i}$, which yields

$$\liminf_{i\to\infty} d_0(\partial^* \phi_{r_i}(x_i)) \geq \sigma > 0. \tag{3.5}$$

Case 4. $d_{X^{-r_i}}(x_i) > r_i$ and $f_{r_i}(x_i) = c$.

The Clarke gradient can be computed in a set of density 1 at x_i [11]. Since $\nabla f_{r_i}(x_i)$ is non zero, the set $\{y \mid f_{r_i}(y) \neq c\}$ has density 1 at x. Now without loss of generality by extracting we can assume x_i converges to a x in $\partial X \cap f^{-1}(c)$ and we obtain

$$\liminf_{i\to\infty} d_0(\partial^* \phi_{r_i}(x_i)) \geq d_0(A_x) \geq \sigma > 0. \tag{3.6}$$

Case 5. $x_i \in \partial X^{-r_i}$ and $f_{r_i}(x_i) > c$.

If $r_i > 0$, then $\partial^* d_{X^{-r_i}}(x_i)$ is the convex set generated by 0 and the direction normal to X^{-r_i} at x_i, that is$[0,1] \cdot \nu(\xi_{\neg X}(x_i) - x_i)$. Note that this direction belongs in the normal cone $N(X, \xi_{\neg X}(x))$. Adding the contribution of f_{r_i} we obtain

$$\partial^* \phi_{r_i}(x_i) \subset A_{\xi_{\neg X}(x_i)}.$$

If $r_i = 0$, then $\partial^* \phi_{r_i}(x_i) = [0,1] \cdot \partial^* d_X(x_i) + \nabla f_{r_i}(x_i)$ and we obtain

$$\partial^* \phi_{r_i}(x_i) \subset A_{x_i}.$$

Either way, $\liminf_{i\to\infty} d_0(\partial^* \phi_{r_i}(x_i)) \geq \sigma > 0$.

Now the remaining cases fit inside the sets of x such that $0 < d_{X^{-r}}(x) \leq r$. Remark that $\text{reach}(X^{-r}) \geq r$. If $d_{X^{-r}}(x) < r$ we know that x has only one closest point $\xi_{X^{-r}}(x)$ in X.

$$\partial^* d_{X^{-r}}(x) = \{\nu(x - \xi_X(x))\}$$

If $d_{X^{-r}}(x) = r$, x belongs to ∂X and the Clarke gradient $\partial^* d_{X^{-r}}(x)$ is $\text{Conv}(\text{Nor}(X,x) \cap \mathbb{S}^{d-1})$ which is $\text{Conv}(\text{Cone}\, \partial^* d_X(x) \cap \mathbb{S}^{d-1})$ by Theorem 2.9. These considerations are illustrated in Figure 8 with $0 < d_{X^{-r}}(x_1) < r$ and $d_{X^{-r}}(x_2) = r$. In any case, this leads to $\partial^* d_{X^{-r}}(x) \subset \partial^* d_X(\xi_{\neg X}(x))$.

Case 6. $0 < d_X^{-r_i}(x_i) \leq r_i$ and $f_{r_i}(x_i) \geq c$

$\partial^* \phi_{r_i}(x_i) \subset \text{Conv}\left(\text{Nor}(X, \xi_{\neg X}(x)) \cap \mathbb{S}^{d-1}\right) + [0,1] \cdot \nabla f_{r_i}(x_i)$. Now by compactness assume that $x_i \to x$. Then $x \in \partial X \cap f^{-1}(c)$ and thus

$$\liminf_{i\to\infty} d_0(\partial^* \phi_{r_i}(x_i)) \geq \sigma > 0. \tag{3.7}$$

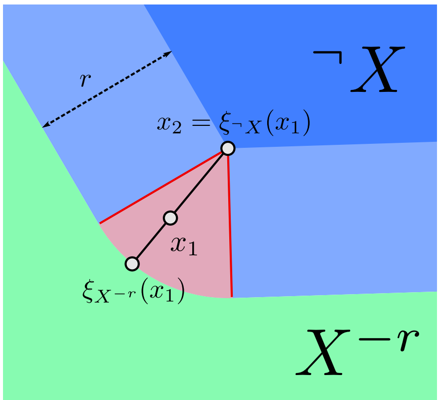


FIGURE 8: Visualisation of the inclusion $\partial^* d_{X^{-r}}(x) \subset \partial^* d_X(\xi_{\neg X}(x))$ for two points x_1 and x_2. The translated unit cone $x_2 + \mathrm{Nor}(\neg X, x_2) \cap B(x_2, r)$ is depicted in red.

Case 7. $0 < d_X^{-r}(x_i) \le r_i$ and $f_{r_i}(x_i) < c$

Then $\partial^* \phi_{r_i}(x_i) \subset \mathrm{Conv}\left(\partial^* d_X(\xi_{\neg X}(x_i)) \cap \mathbb{S}^{d-1}\right)$ which yields

$$\liminf_{i\to\infty} d_0(\partial^* \phi_{r_i}(x_i)) \ge \mu > 0. \tag{3.8}$$

□

3.3 Constant homotopy type Lemma

In this subsection we prove that under our assumptions the topology of the sublevel sets does not evolve in between critical values.

Theorem 3.6: Constant homotopy type in between critical values

Let $X \subset \mathbb{R}^d$ be a complementary regular set. Let $f : \mathbb{R}^d \to \mathbb{R}$ and $a < b \in \mathbb{R}$ be such that $[a, b]$ contains only regular values of $f_{|X}$.

Then X_a is a deformation retraction of X_b.

This theorem is a direct consequence of Lemma 3.8. We prove a technical lemma first.

Lemma 3.7: Locally non-vanishing Clarke gradients

Let c be a regular value of $f_{|X}$.
Then

$$\lim_{\substack{\varepsilon\to 0^+ \\ K\to 0^+}} \inf\left\{d_0(\partial^* \phi_{c+a}(x)), x \in \phi_{c+a}^{-1}(0, K] \mid a \in [-\varepsilon, \varepsilon]\right\} > 0$$

Proof. We proceed by contradiction. Assuming the inequality is false, there exist two sequences in $\mathbb{R}$ $a_i \to 0, K_i \to 0^+$ and $(x_i)_{i\in\mathbb{N}}$ a sequence in $\mathbb{R}^d$ such that

$$\lim_{i\to\infty} d_0(\partial^* \phi_{c+a_i}(x_i)) = 0$$

We keep the partition of $\phi_{c+a}^{-1}(0, K)$ as in the proof of Lemma 3.5. With $r = 0$, we obtain 5 cases to compute $\partial^* \phi_{c+a_i}$.

Case 1. $f(x_i) < c + a_i$. Then $\partial^* \phi_{c+a_i}(x_i) = \partial^* d_X(x_i)$ and thus

$$\liminf_{i\to\infty} d_0(\partial^* \phi_{c+a_i}(x_i)) \geq \mu > 0$$

Case 2. $x_i \in \mathrm{int}(X)$. Then $\partial^* \phi_{c+a_i}(x_i) = \{\nabla f(x_i)\}$ and thus

$$\liminf_{i\to\infty} d_0(\partial^* \phi_{c+a_i}(x_i)) \geq \sigma > 0$$

Cases 3, 4, 5.

$$\begin{cases} f(x_i) > c + a_i & \text{and} & d_X(x_i) > 0 \\ f(x_i) > c + a_i & \text{and} & x_i \in \partial X \\ f(x_i) = c + a_i & \text{and} & d_X(x_i) > 0 \end{cases}$$

In these 3 cases we have the inclusion $\partial^* \phi_{c+a_i}(x_i) \subset A_{x_i}$. As in the proof of Lemma 3.5, the map $y \mapsto A_y$ is lower semicontinuous. Now if (x_i) converges to x then it belongs to $\partial X \cap f^{-1}(c)$ and c being a regular value yields

$$\liminf_{i\to\infty} d_0(\partial^* \phi_{c+a_i}(x_i)) \geq \kappa > 0$$

□

Lemma 3.8: Local deformation retractions

Let X be complementary regular, $f : \mathbb{R}^d \to \mathbb{R}$ smooth and let c be a regular value of $f_{|X}$. Then $X_{c-\varepsilon}$ is a deformation retraction of X_{c+a} for all $a \in [-\varepsilon, \varepsilon]$ for any $\varepsilon > 0$ small enough.

Proof. Put σ the positive constant obtained in Lemma 3.7. Thus for every $a \in [-\varepsilon, \varepsilon]$ there exists a $\frac{2K}{\sigma}$-Lipschitz approximate flow of ϕ_{c+a} on $\phi_{c+a}^{-1}(0, K]$ which we will denote $C_{c+a}(\cdot, \cdot)$. We also fix $M > 0$ such that the ϕ_{c+a} are all M Lipschitz over the sets we are considering.

Thus $C_{c-\varepsilon}$ is well-defined at any time in $X_{c-\varepsilon}^{\frac{K}{M}} \subset \phi_{c-\varepsilon}[0, K]$. Now since $(\phi_{c+a})_{a\in[-\varepsilon,\varepsilon]}$ is a family of Lipschitz functions whose constants are uniformly bounded, there is a constant $Q > 0$ such that $X_{c+a} \subset X_{c-\varepsilon}^{\varepsilon Q}$ for all $a \in [-\varepsilon, \varepsilon]$. For ε small enough we also have $X_{c+a} \subset \phi_{c-\varepsilon}^{-1}[0, K]$. Thus the approximate flow $C_{c-\varepsilon}(\cdot, \cdot)$ restricted to $[0, 1] \times X_{c+a}$ is well-defined for any $a \in [-\varepsilon, \varepsilon]$ when $\varepsilon > 0$ is small enough.

Now we show that for $\varepsilon > 0$ small enough, for any $a \in [-\varepsilon, \varepsilon]$ the end-flow $C_{c-\varepsilon}(1, \cdot)_{|X_{c+a}}$ is homotopic to $\mathrm{Id}_{X_{c+a}}$ via the homotopy

$$(t, x) \mapsto C_{c+a}(1, C_{c-\varepsilon}(t, x))$$

The homotopy is well-defined at any point as $C_{c-\varepsilon}$ is $\frac{2K}{\sigma}$-Lipschitz in time parameter, yielding $C_{c-\varepsilon}(1, X_{c+a}) \subset X_{c+a}^{\frac{2K\varepsilon}{\sigma}}$. This last set is a subset of $\phi_{c+a}^{-1}[0, K]$ when $\varepsilon > 0$ is small enough. □

3.4 Handle attachment around critical values

First, we describe how a cell is glued around a unique critical point.

Proposition 3.9: Around unique critical values

Let X be complementary regular and $f : \mathbb{R}^d \to \mathbb{R}$. Assume $f_{|X}$ has only one critical point p in $f^{-1}(c)$ which is non degenerate.
Then for any $\varepsilon > 0$ small enough $X_{c+\varepsilon}$ has the homotopy type of $X_{c-\varepsilon}$ attached with a

λ-cell, where

$$\lambda_p := \textit{ indice of the Hessian at } p + \textit{ number of infinite curvatures at } p$$

Proof. Let x_c be the sole critical point with value c and $n_c = \frac{\nabla f(x_c)}{||\nabla f(x_c)||} \in \mathrm{Nor}(^\neg X, x_c)$ the normalized gradient of f at this point. Put $f_r(x) = f(x - rn_c)$ to be f translated in the direction n_c with magnitude r.

The pair $(x_c, n_c) \in \mathrm{Nor}(^\neg X)$ is regular by non-degeneracy of f at x. Denote $(\kappa_i')_{1\leq i\leq d-1}$ the principal curvatures (cf. Proposition 2.4) of $^\neg X$ at (x_c, n_c) sorted in ascending order and put $m = \max\{i, \kappa_i' < \infty\}$. The regularity of (x_c, n_c) for X guarantees that the Gauss map $x \in \partial^\neg X^{-r} \mapsto n(x) \in \mathbb{S}^{d-1}$ is differentiable at $x_c + rn_c$. The principal curvatures of $^\neg X^{-r}$ at $(x_c + rn_c, n_c)$ can be obtained from the κ_i' via $\kappa_{i,r}' = \frac{\kappa_i'}{1+r\kappa_i'}$. Note that when a principal curvature κ_i' is infinite, the previous equality is valid with $\kappa_{i,r}' = \frac{1}{r}$.

The Gauss map of X^{-r} is the opposite of the previous one and thus also differentiable at $x_c + rn_c$. The principal curvatures $(\kappa_{i,r})_{1\leq i\leq d-1}$ of X^{-r} at $(x_c + rn_c, -n_c)$ are the opposite of that of $^\neg X^{-r}$ $\kappa_{i,r} = -\kappa_{i,r}'$.

Let $a, b \in \mathrm{Tan}(X^{-r}, x_c + rn_c)$. The Hessian $H_r f_r$ of $(f_r)_{|X^{-r}}$ at $x_c + rn_c$ (cf. Definition 2.5) is exactly

$$H_r f_r(a, b) = H f_r(a, b) + ||\nabla f_r(x_c + rn_c)||\, \mathrm{II}_r(x_c + rn_c)(a, b)$$

where $\mathrm{II}_r(x_c + rn_c)$ is the second fundamental form of X^{-r} at $x_c + rn_c$. Proceeding exactly in the same fashion as the proof of 4.6 in [3] we obtain that there exist matrices A_1, A_2, A_3, C, B such that in a good basis the Hessian $H_r f_r$ has the form

$$\begin{pmatrix} A_1 + rA_2 + r^2 A_3 & rC \\ rC^t & -r\, ||\nabla f(p)||\, I_d + r^2 B \end{pmatrix}$$

where A_1 is the diagonal matrix of dimension m with diagonal $(-\kappa_i')_{1\leq i\leq m}$. It is the same computation as [3] except that we end up with a minus sign in front of the identity in the lower right corner. When $r > 0$ is small enough, the index of this matrix is that of A_1 plus the dimension of the identity matrix in the lower right corner. Then, we apply classical Morse Theory on sets bounded by a $C^{1,1}$ hypersurface to get the change in topology between $X^{-r}_{c-\varepsilon}$ and $X^{-r}_{c+\varepsilon}$. This is summarized in the following diagram.

□

3.5 Multi-handle attachement

Now we want to understand the change in topology when a critical value might have several corresponding critical points. We begin by showing that non-degenerate critical points of $f_{|X}$ have to be isolated.

Lemma 3.10: Correspondance between critical points of $f_{|X}$ and $f^r_{|X^{-r}}$

Let X be a subset of $\mathbb{R}^d$ and r such that $\mathrm{reach}(^\neg X) > r > 0$. Assume x is a non-degenerate critical point of $f_{|X}$.
Then $x^r = x + r\frac{\nabla f(x)}{||\nabla f(x)||}$ is a critical point of $f^r_{|X^{-r}}$ of the same value.
As a consequence, any non-degenerate critical point of $f_{|X}$ is isolated.

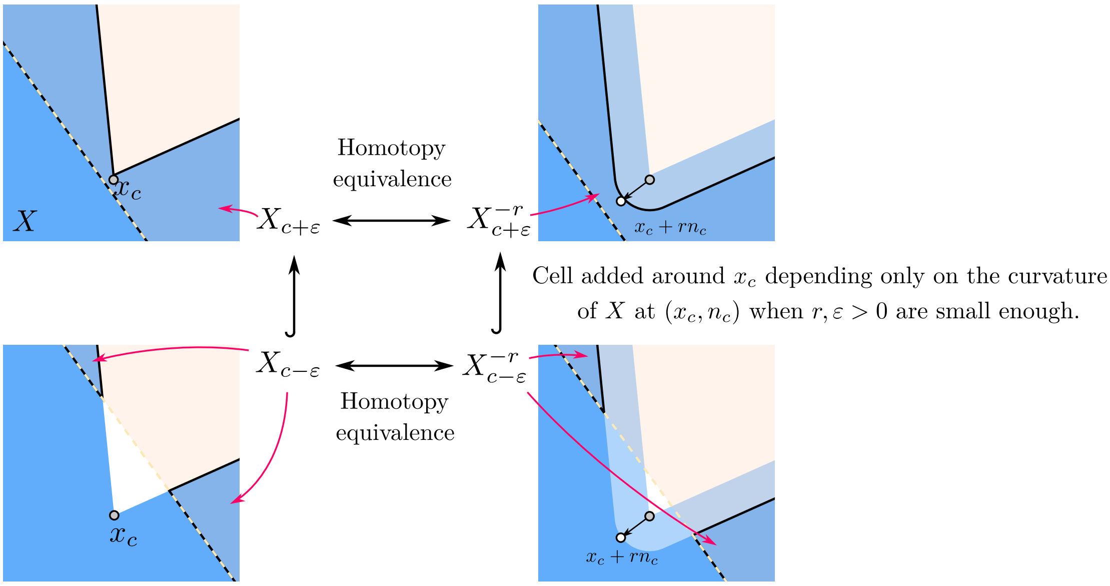


FIGURE 9: Commutative diagram in the proof of Proposition 3.9

Proof. x^r being a critical point of $f^r_{X^{-r}}$ comes from a straightforward computation: we have $f^r(x^r) = f(x)$ and $\nabla f(x) = \nabla f^r(x^r)$, and we know that $\mathrm{Nor}(X^{-r}, x^r) = \mathrm{Cone}(\nabla f(x))$. The last part follows from the isolatedness of critical points in X^{-r}. By the proof of Proposition 3.9, x^r has to be a non-degenerate critical point for $f^r_{|X^{-r}}$ when $r > 0$ is small enough. Any non-degenerate critical point of a $C^{1,1}$ hypersurface has to be isolated. This forces x to be an isolated critical point by continuity of $y \mapsto y + rn_c$. $\square$

Theorem 3.11: Morse Theory for sets whose complement set has positive reach

Let $X \subset \mathbb{R}^d$ and $\mu \in (0, 1]$ such that $\mathrm{reach}_\mu(X) > 0$ and $\mathrm{reach}(^\neg X) > 0$. Suppose $f_{|X}$ has only non-degenerate critical points. Each critical level set $X \cap f^{-1}(\{c\})$ has a finite number p_c of critical points, whose indices (defined in Proposition 3.9) we denote $\lambda_1^c, \dots \lambda_{p_c}^c$. Then

- *If $[a, b]$ does not contain any critical value, X_a is a deformation retract of X_b.*
- *If c is a critical value, $X_{c+\varepsilon}$ has the homotopy type of $X_{c-\varepsilon}$ with exactly p_c cells attached around the critical points in $f^{-1}(c) \cap X$, of respective dimension $\lambda_{p_1}^c, \dots, \lambda_{p_c}^c$ for all $\varepsilon > 0$ small enough.*

Proof. By Lemma 3.10 we know that the critical points in $f_{|X}$ have to be isolated. Put $x_1, \dots, x_p$ the critical points of $f_{|X}$ inside $f^{-1}(c)$. Put $n_i = \frac{\nabla f(x_i)}{||f(x_i)||}$ and $x_i^r = x_i + rn_i$. Let $n(x)$ be the normal n_i associated to the closest critical point x_i of x. This map is piecewise constant and defined almost everywhere. We will show that $\{x_1^r, \dots x_p^r\}$ is exactly the set of critical point of a certain $f^r_{|X^{-r}}$ with f^r a new function built in the following paragraphs.

Let $U_i \subset V_i$ be respectively closed and open balls containing x_i such that $\overline{V_i} \cap \overline{V_j} = \emptyset$ when $j \neq i$.

Let η_c be smooth function on $\mathbb{R}^d$ with values in $[0, 1]$ such that η_c is constant of value 1 inside each U_i and 0 outside of $\bigcup V_i$. The map $\gamma_c : y \mapsto \eta_c(y)n(y)$ is well-defined and continuous when the U_i are small enough. When r is small enough, it is a diffeomorphism.

Finally, we keep the definition $X_c^{-r} = X^{-r} \cap f_r^{-1}(-\infty, c]$ but define a new f_r, which is f

locally translated around the critical points:

$$f_r : x \mapsto f(x + r\gamma_c(x))$$

From Lemma 3.10 we know that the $(x_i^r)_{1\leq i\leq p}$ are non-degenerate critical point of X^{-r} for $f_{r|X^{-r}}$ with corresponding index $(\lambda_i^c)_{1\leq i\leq p}$. From Lemma 4.8 in [3], we know that x_i^r is the only critical point of $f_{r|X^{-r}}$ inside $f_r(U_i)$ when r is small enough.

Now we prove that there are no critical points outside of $\cup_i f_r(U_i)$ when r is small enough. By classical theorems X^{-r} has a $C^{1,1}$ boundary. Since ∇f does not vanish in a neighborhood of $f^{-1}(c) \cap X$, we know that $x \in X^{-r}$ is a critical point of $f_{r|X^{-r}}$ if and only if $x \in \partial X^{-r}$, $\{\nu\} = \mathrm{Nor}(X^{-r}, x) \cap \mathbb{S}^{d-1}$ (i.e ν is *the* normal at x) and $\left|\left|\frac{\nabla f_r(x)}{||\nabla f_r(x)||} - \nu\right|\right| = 0$.

Remark that we have both

- $\mathrm{Nor}(X^{-r}) = \{(x + r\nu, -\nu) \mid (x, \nu) \in \mathrm{Nor}(^{\neg}X)\}$
- $\sup_{(x,\nu)\in \mathrm{Nor}(X)} ||\nabla f(x) - \nabla f_r(x + r\nu)|| = O(r)$

leading to

$$\liminf_{r\to 0} \inf_{\substack{(x,\nu)\in \mathrm{Nor}(X^{-r}) \\ x\notin \cup_i f_r(U_i) \\ f_r(x)=c}} \left|\left|\frac{\nabla f_r(x)}{||\nabla f_r(x)||} - \nu\right|\right| \geq \inf_{\substack{(x,\nu)\in \mathrm{Nor}(^{\neg}X) \\ x\notin \cup_i U_i \\ f(x)=c}} \left|\left|\frac{\nabla f(x)}{||\nabla f(x)||} - \nu\right|\right| > 0 \tag{3.9}$$

Thereby showing that $\{x_1^r, \dots, x_p^r\}$ is exactly the set of critical points of $f_{r|X^{-r}}$ with value c. We obtain $X_{c+\varepsilon}^{-r}$ from $X_{c-\varepsilon}^{-r}$ by gluing cells locally around each critical point as in classical Morse Theory. □

Remark. A similar argument holds assuming X has positive reach, thereby showing that Morse Theorems are still true when X has positive reach and f is a Morse function with several non-degenerate critical points sharing the same critical value, by taking $\eta_c(x)$ to be -1 near critical points instead of 1.

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