Approximating the area of a set: when geometric measure theory meets persistent homology.

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Motivation.

Let $Y \subset \mathbb{R}^d$ be an approximation of $X \subset \mathbb{R}^d$.

→ How to approximate $\mathcal{H}^{d-1}(\partial X)$ (boundary area) from the approximating set Y?

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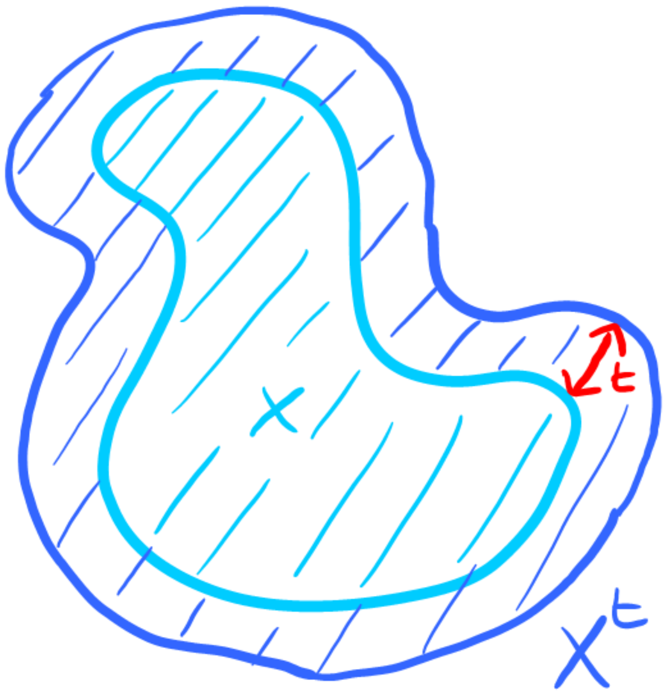
Today, we talk about

- Issues when X is not smooth
- More general family of intrinsic volumes
- Approach to answer this problem using persistence

Offsets.

Let $X, Y \subset \mathbb{R}^d$



Distance to X

$$X^t := \{x \in \mathbb{R}^d, d_X(x) \leq t\}$$

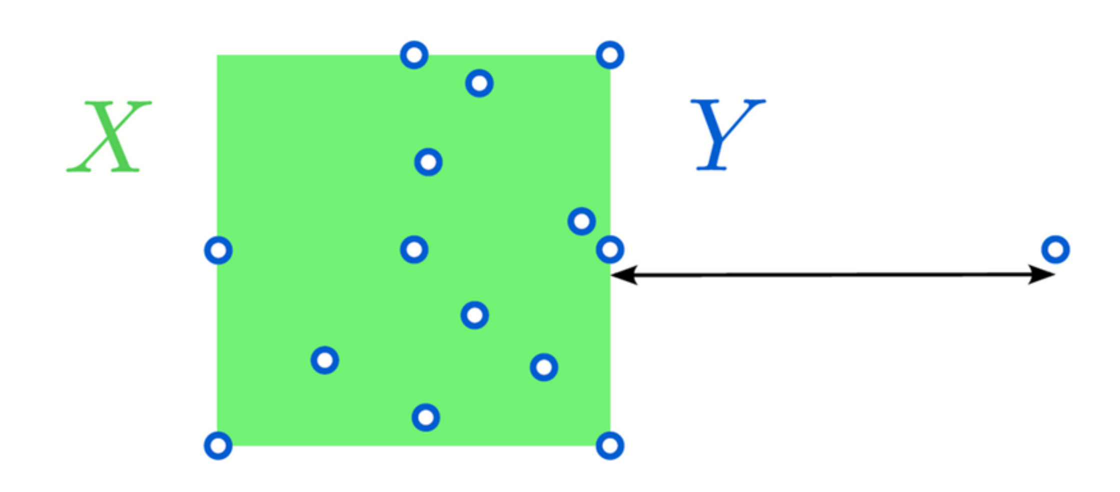
is the t-offset of X.

Hausdorff distance.

Let $X, Y \subset \mathbb{R}^d$

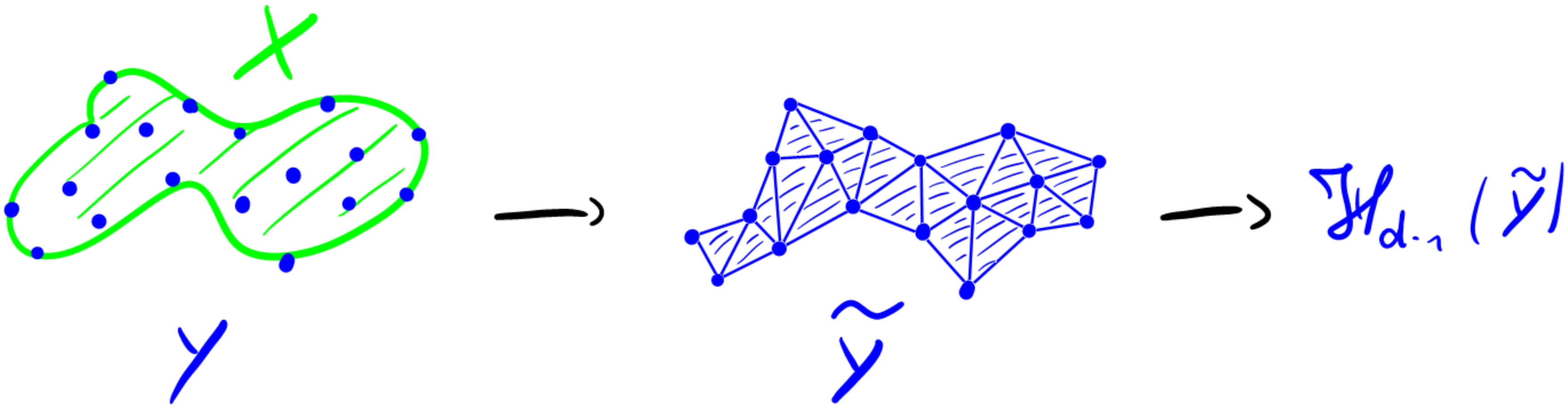
We use the Hausdorff distance between X and Y

$$d_H(X,Y) = \inf\{t \geq 0,\ X \subset Y^t,\ Y \subset X^t\}$$



Inference in the smooth case

When X is smooth, the boundary area $\mathcal{H}_{d-1}(\partial X)$ can be recovered by triangulating Y.



Such methods are supported by a vast literature.

Inference in the non-smooth case

When X is not smooth,

classical methods fail to reconstruct X.

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classical methods fail to reconstruct X.

The only known method to reconstruct X in a faithful manner consists of using offsets X^ε. This recovers the homotopy type of X. (Chazal et al., 2007).

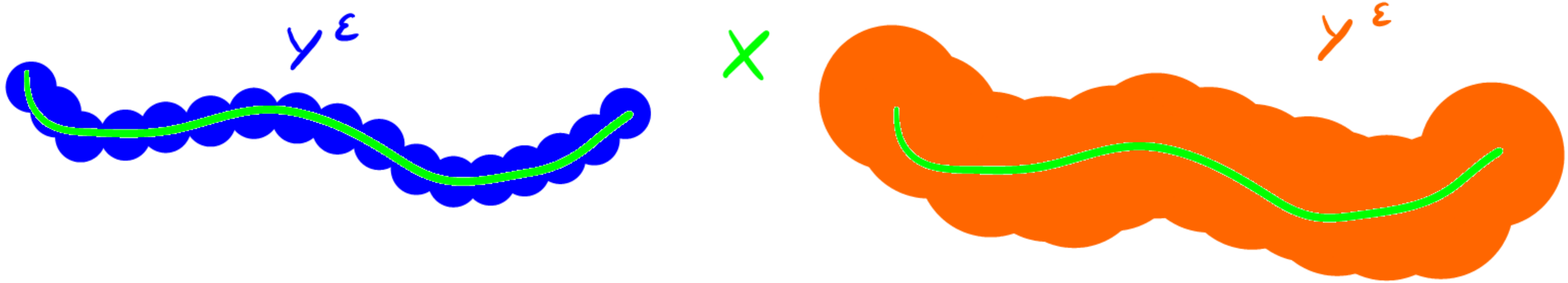
Inference in the non-smooth case

When X is not smooth,

classical methods fail to reconstruct X..

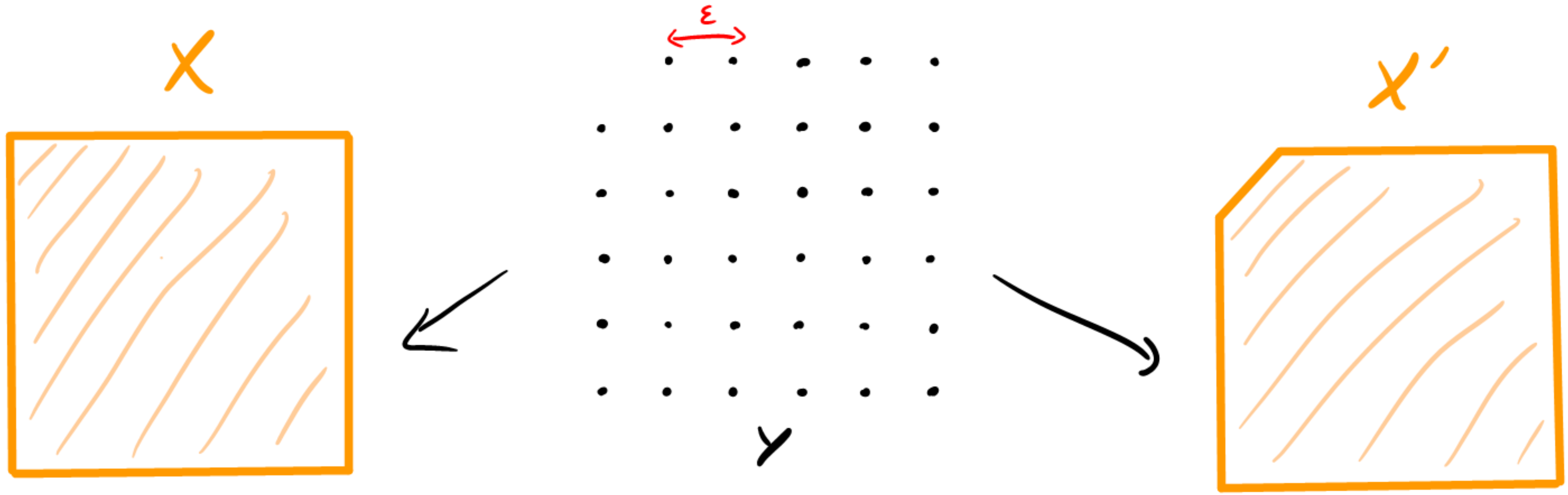
$\mathcal{H}_{d-1}(\partial Y^{\varepsilon})$ fail to properly converge to $\mathcal{H}_{d-1}(X)$ as either

- Y^{ε} is too noisy (ε is too small)
- Y^{ε} is smoother but far from X (ε is too large)



Inference in the non-smooth case

In the non-smooth case, the general rate of convergence with respect to the Hausdorff distance cannot be better than linear.



$$d_H(X, Y), d_H(X', Y) = O(\varepsilon)$$

but

$$\varepsilon = O(\mathcal{H}_{d-1}(\partial X) - \mathcal{H}_{d-1}(\partial X'))$$

Intrinsic Volumes

the boundary area $\mathcal{H}^{d-1}(\partial X)$ is part of the family of intrinsic volumes.

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→ Quantities $V_0(X), V_1(X), \dots, V_{d-1}(X), V_d(X)$

Intrinsic Volumes

the boundary area $\mathcal{H}^{d-1}(\partial X)$ is part of the family of intrinsic volumes.

$\longrightarrow$ Quantities $V_0(X), V_1(X), \ldots, V_{d-1}(X), V_d(X)$

Volume of X.

Intrinsic Volumes

→ Quantities $V_0(X), V_1(X), \dots, V_{d-1}(X), \cancel{V_d(X)}$

are related to the curvatures of X.

Intrinsic Volumes

→ Quantities $V_0(X), V_1(X), \dots, V_{d-1}(X), \cancel{V_d(X)}$ related to the curvatures of X.

→ If X domain bounded by a hypersurface :

Let $\kappa_1, \dots, \kappa_{d-1}$ be the principal curvatures at $x \in \partial X$

= eigenvalues of differential of $x \longmapsto n(x)$

"Gauss map".

Intrinsic Volumes

→ Quantities $V_0(X), V_1(X), \dots, V_{d-1}(X)$ related to the curvatures of X.

→ If X domain bounded by a hypersurface :

$$V_i(X) = \omega_{d-i}^{-1} \int_{\partial X} \Big(\sum_{1 \le i_1 < \dots < i_{d-i-1} \le d-1} \kappa_{i_1} \cdots \kappa_{i_{d-i-1}} \Big) \, d\omega_X(x)$$

Constants depending on d, i.

Symmetric homogeneous polynomial of degree $(d-i-1)$ in the principal curvatures.

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Particular case :

$$V_0(X) = \omega_d^{-1} \int_{\partial X} \kappa_1 \cdots \kappa_{d-1} \, d\omega_X(x) = \chi(X)$$

(Gauss-Bonnet)

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Particular case :

$$V_{d-2}(X) = \omega_2^{-1} \int_{\partial X} (\kappa_1 + \cdots + \kappa_{d-1}) \, d\omega_X(x) \quad \text{(Mean Curvature)}$$

Intrinsic Volumes

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Particular case:

$$V_{d-1}(X) = \omega_1^{-1} \int_{\partial X} 1 \, d\omega_X(x) \qquad \text{(Boundary Area)}$$

Intrinsic Volumes

More general definition using the reach

$$\mathrm{reach}(X) := \sup \{ t \geq 0,\ 0 < d_X(x) < t \Rightarrow x \text{ has a unique closest point in } X \}$$

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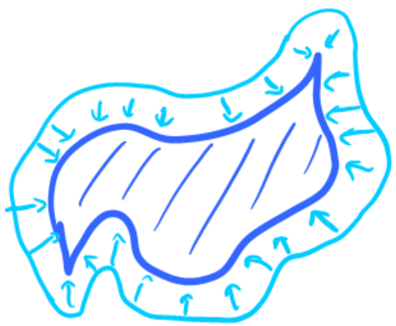
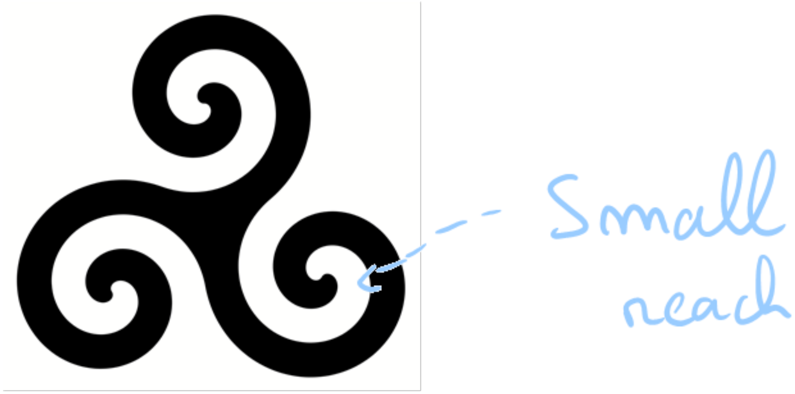
More general definition using the reach

$$\mathrm{reach}(X) := \sup \{ t \geq 0,\ 0 < d_X(x) < t \Rightarrow x \text{ has a unique closest point in } X \}$$

<u>Idea</u>: Positive reach $\simeq$ locally smooth or convex.

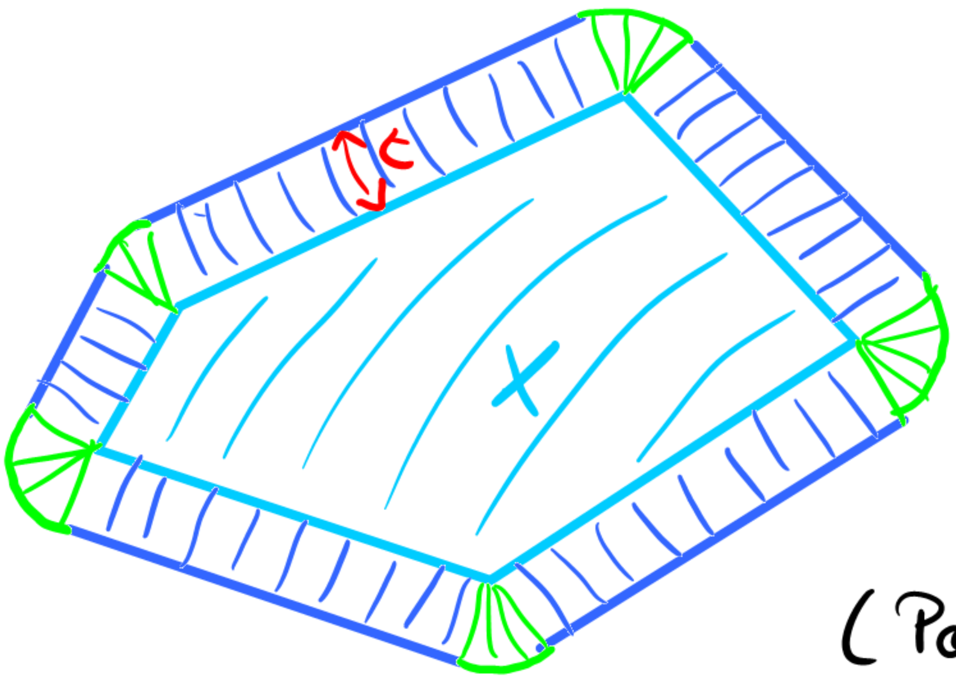




Tube formula (Federer, 1959)

Within $[0, \mathrm{reach}(X)]$,

$t \mapsto \mathrm{Vol}(X^t)$ is a polynomial.



$$\mathrm{Vol}(X^t) = \mathrm{Vol}(X) + \mathrm{length}(\partial X)\, t + 2\pi\, \chi(X)\, t^2$$

(Polyhedra: Steiner, 1842)

Principal kinematic formula

(Chern, Santalo)

<u>Theorem</u>: for every non-pathological set X you can think of, for $t \geq 0$:

$$\int_{\mathbb{R}^d} \chi(X \cap \underbrace{B(x,t)}_{\text{Ball of radius } t}) \, dx = \underbrace{\sum_{i=0}^{d} \omega_i t^i V_{d-i}(X)}_{\text{Steiner polynomial of } X}$$

Principal kinematic formula

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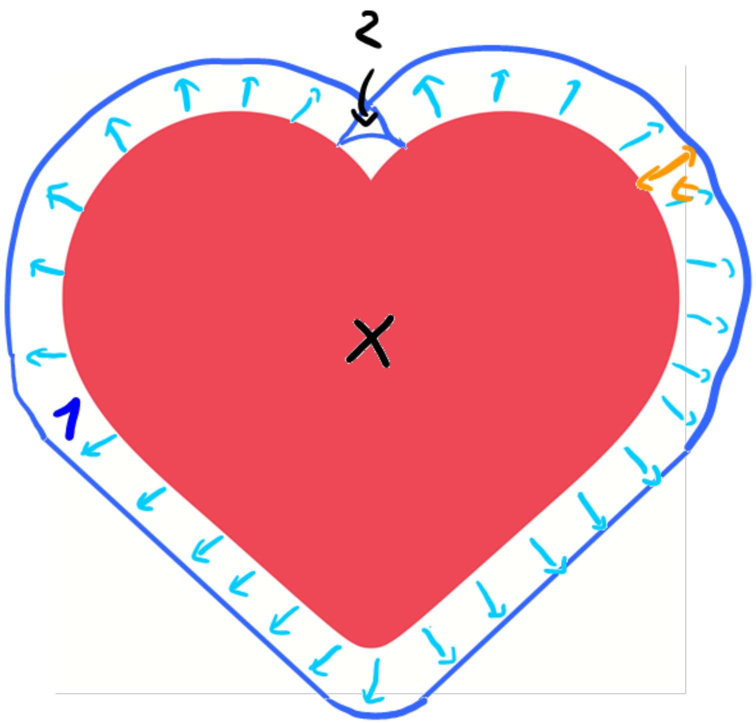
Example:

Integration over X^t



Principal kinematic formula.

Integrating the formula over t yields

$$\int_{\mathbb{R}^d}\int_0^R \chi(X\cap B(x,t))\,dt\,dx = Q_X(R)$$

Polynomial in $V_i(X)$

Principal kinematic formula.

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Polynomial in $V_i(X)$

- $\forall x \in \mathbb{R}^d$, $(X \cap B(x,t))_{t \in \mathbb{R}}$ is a filtration

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Polynomial in $V_i(X)$

- $\forall x\in\mathbb{R}^d$, $(X\cap B(x,t))_{t\in\mathbb{R}}$ is a filtration
- The Euler characteristic is a topological quantity.

(alternating sum of Betti numbers)

Inviting us to use persistent homology.

Persistent homology.

Analysis of the topological evolution of a filtration

$(X_t)_{t \in \mathbb{R}}$

$s \leq t \Rightarrow X_s \subset X_t$

Persistent homology.

Analysis of the topological evolution of a filtration

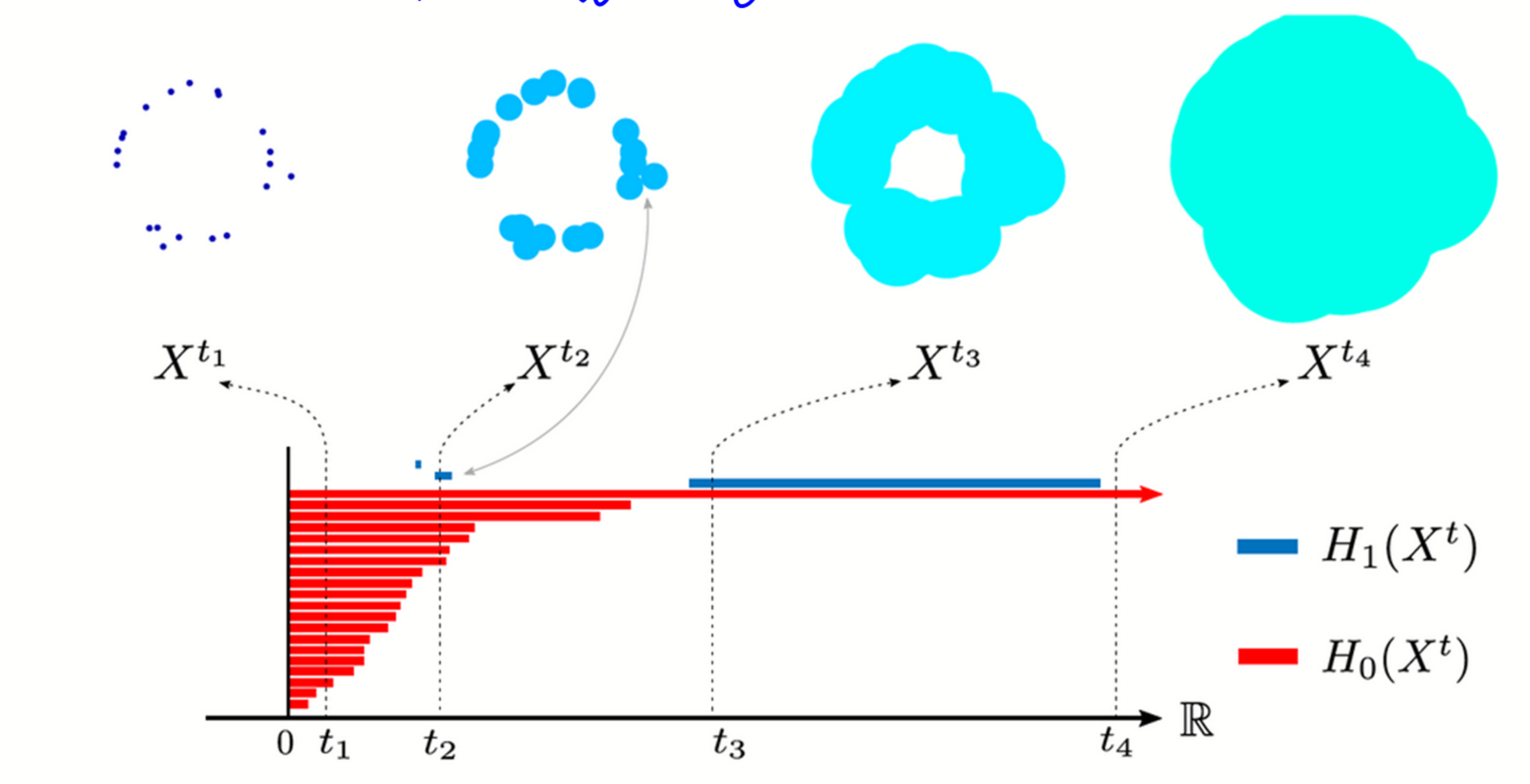
$$(X_t)_{t \in \mathbb{R}}$$

via the family $(H_i(X_t))_{t \in \mathbb{R}}$

(Recall $\dim H_i(Z)$ is interpreted as the number of i-dimensional features (or voids) of Z)

Persistent homology.

Example: offset filtration.



Graded multiset of bars: persistent homology diagrams

Persistent idea.

- $(X \cap B(x,t))_{t \in \mathbb{R}^+}$ = sublevel set filtration of $(d_x)_{|X} : y \longmapsto \|x-y\|$

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- $(X \cap B(x,t))_{t \in \mathbb{R}^+}$ = sublevel set filtration of $(d_x)_{|X} : y \longmapsto \|x-y\|$

- Associate a persistent diagram $\mathrm{dgm}(d_{x|X})$ by studying $H_i (X \cap B(x,t))_{t \in \mathbb{R}^+}$. $0 \leq i \leq d$.

Persistent idea.

- $(X \cap B(x,t))_{t \in \mathbb{R}^+}$ = sublevel set filtration of $(d_x)_{|X} : y \longmapsto \|x-y\|$
- Associate a persistent diagram $\mathrm{dgm}(d_{x|X})$ by studying $H_i(X \cap B(x,t))_{t\in\mathbb{R}^+}$, $0 \le i \le d$.
- $\chi(X \cap B(x,t)) =$ Alternating sum of intervals of $\mathrm{dgm}(d_{x|X})$ containing t.

Idea Sketch.

- From Y, construct a family of diagrams

$$(D_x^Y)_{x \in \mathbb{R}^d}$$

such that $d_B(\mathrm{dgm}(d_{x|X}), D_x^Y) \leq \varepsilon$

metrics on diagrams

$d_H(X, Y)$.

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metrics on diagrams

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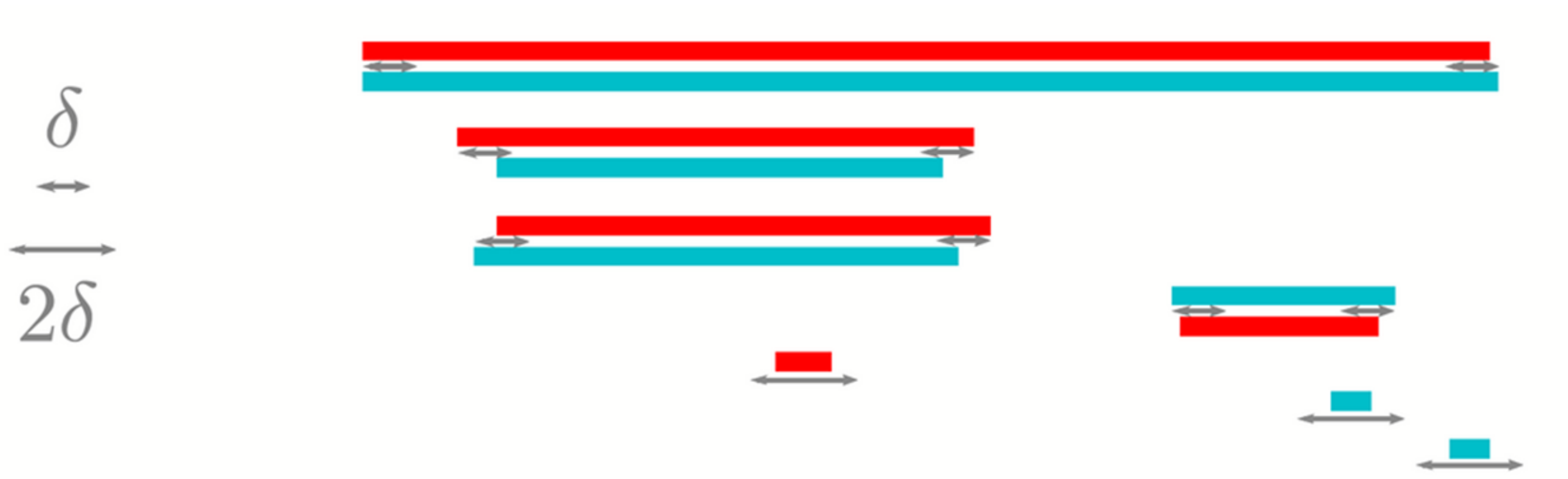
Ex: $d_B(\mathrm{Dgm}_1, \mathrm{Dgm}_2) \leq \delta \Rightarrow$ partial bijection as follows



Idea Sketch.

- From Y, construct a family of diagrams

$$(D_x^Y)_{x \in \mathbb{R}^d}$$

such that $d_B(\mathrm{dgm}(d_{x|X}), D_x^Y) \leq \varepsilon$

- Technical lemma:

$$\int_0^R |\chi(D_x^Y, t) - \chi(\mathrm{dgm}(d_{x|X}), t)| \leq \varepsilon \times (N(D_x^Y, t) + N(\mathrm{dgm}(d_{x|X})))$$

Total number of bars meeting $[0, t]$ in each diagram.

Idea Sketch.

Consequence: Let

$$\hat{Q}_Y(R) := \int_{\mathbb{R}^d}\int_0^R \chi(D_x^Y, t)\, dt\, dx$$

Replacing $\chi(\mathrm{dgm}(d_{x|X}, t))$ by

$\chi(D_x^Y, t)$

in the principal kinematic formula

Idea Sketch.

Consequence: Let

$$\hat{Q}_Y(R) := \int_{\mathbb{R}^d}\int_0^R \chi(D_x^Y, t)\,dt\,dx$$

$$\Rightarrow \quad \| Q_X - \hat{Q}_Y \|_{\infty,[0,R]} \leq \varepsilon \times \int_{\mathbb{R}^d} N(D_x^Y, R) + N(dgm(d_{x|X}, R))\,dx$$

by the previous technical lemma.

Idea Sketch.

Consequence: Let

$$\widehat{Q}_Y(R) := \int_{\mathbb{R}^d}\int_0^R \chi(D_x^Y, t)\,dt\,dx$$

$$\Rightarrow \|Q_X - \widehat{Q}_Y\|_{\infty,[0,R]} \leq \varepsilon \times \int_{\mathbb{R}^d} N(D_x^Y, R) + N(\mathrm{dgm}(d_{x|X}, R))\,dx$$

Fact:

$$\int_{\mathbb{R}^d} N(D_x^Y, R) + N(\mathrm{dgm}(d_{x|X}, R))\,dx \leq K \times TC(X)$$

Total curvatures of X

By Morse theory + Geometric Measure Theory

Conclusion

One can retrieve "coefficients" $\hat{V}_i(Y)$ from $\hat{Q}_Y$.

$$\Downarrow$$

Theorem: C., Cohen-Steiner (2024)

if X has weak regularity*, $\varepsilon \geq d_H(X,Y)$, one can construct $\hat{V}_i(Y)$ such that

$$\boxed{|V_i(X) - \hat{V}_i(Y)| = O(\varepsilon \times TC(X))}$$

and in particular,

$$\boxed{|\mathcal{H}_{d-1}(X) - \hat{V}_{d-1}(Y)| = O(\varepsilon \times TC(X))}$$

Technical Catch:

How to construct D_x^Y ?

Write $Y_t^\varepsilon = Y^\varepsilon \cap B(x,t)$

$$\begin{tikzcd}
\arrow[r, dashed] & H_i(Y^{3\varepsilon}_s) \arrow[r] & H_i(Y^{3\varepsilon}_t) \arrow[r, dashed] & {} \\
\arrow[r, dashed] & H_i(X^{2\varepsilon}_s) \arrow[r] \arrow[u] & H_i(X^{2\varepsilon}_t) \arrow[r, dashed] \arrow[u] & {} \\
\arrow[r, dashed] & H_i(Y^{\varepsilon}_s) \arrow[r] \arrow[u] \arrow[uu, bend right] & H_i(Y^{3\varepsilon}_t) \arrow[r, dashed] \arrow[u] \arrow[uu, bend right] & {}
\end{tikzcd}$$

Image of $H_i(Y^\varepsilon_t)$ in $H_i(Y^{3\varepsilon}_t)$

Final Remark

Corollary of our analysis, answering Milnor:

Theorem (C., Cohen-Steiner)

If there is an homotopy equivalence between X and Y, moving points by less than ε then

$$\boxed{|V_i(X) - V_i(Y)| = O(\varepsilon \times (TC(X) + TC(Y)))}$$

Thank you!