Morse Theory for Tubular Neighborhoods.

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Oxford Applied Topology Seminar

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0) Preamble

I) Classic Morse Theory

II) Morse theory for sets with positive reach

III) Morse theory for Tubular Neighborhoods.

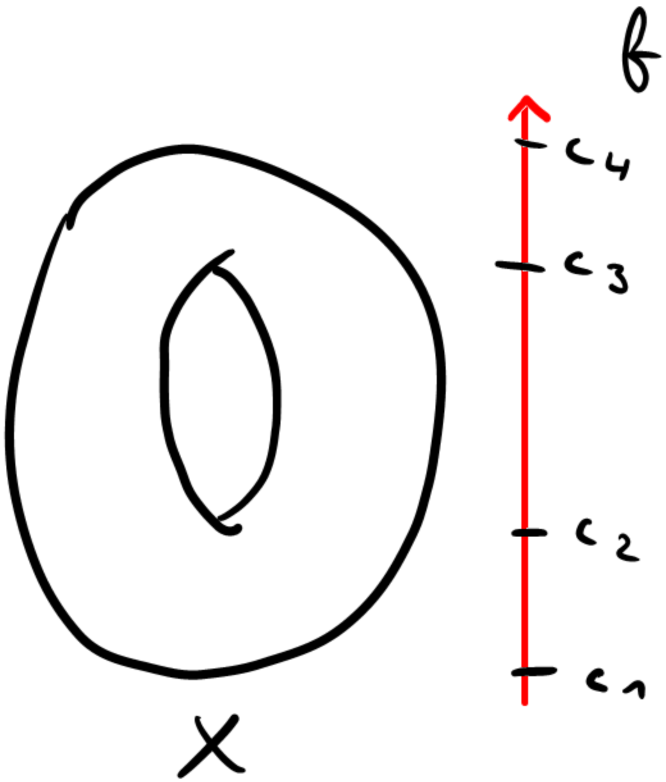
Say $f : X \to \mathbb{R}$ is generic.

Then the topology of $c \mapsto X_c$ should only rarely change.

i.e there exists $\{c_1, \dots, c_m\}$ a finite set of critical values such that

$$\begin{cases} X_s \sim X_t \\ X_{c_i+\varepsilon} \sim X_{c_i-\varepsilon} \text{ with a cell attached.} \end{cases}$$

Example: • X a torus
• f a height function



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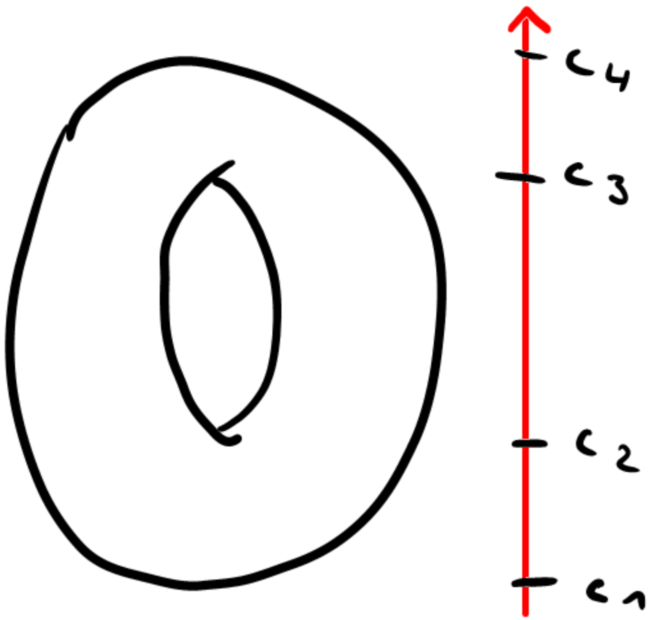

X_c, $c \in (-\infty, c_1)$

$\emptyset$

Example:
- X a torus
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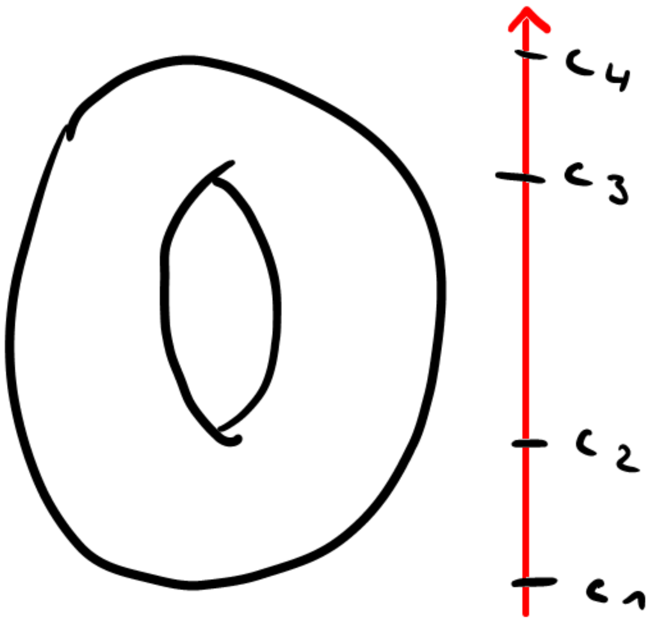

X_c, $c \in (c_1, c_2)$

 $\sim \cdot$

Example:
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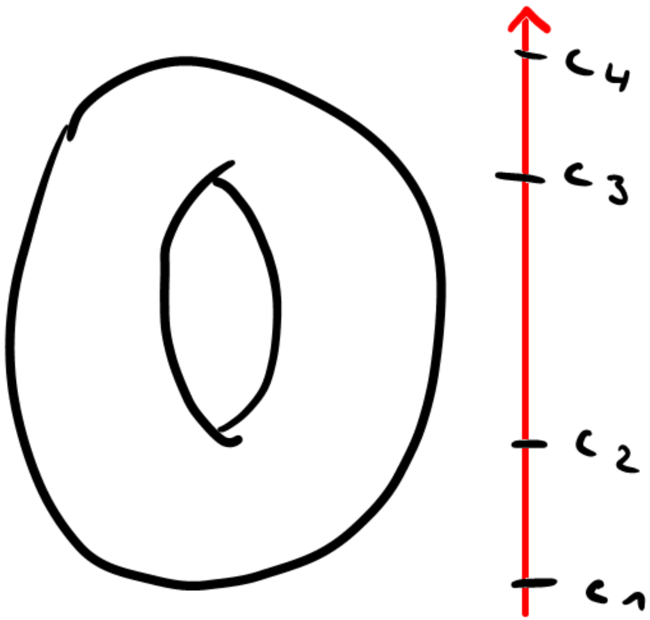
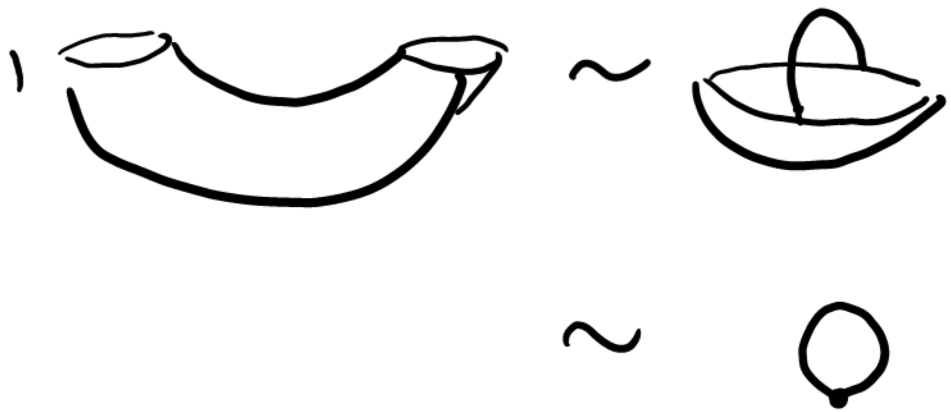


X_c , $c \in (c_2, c_3)$



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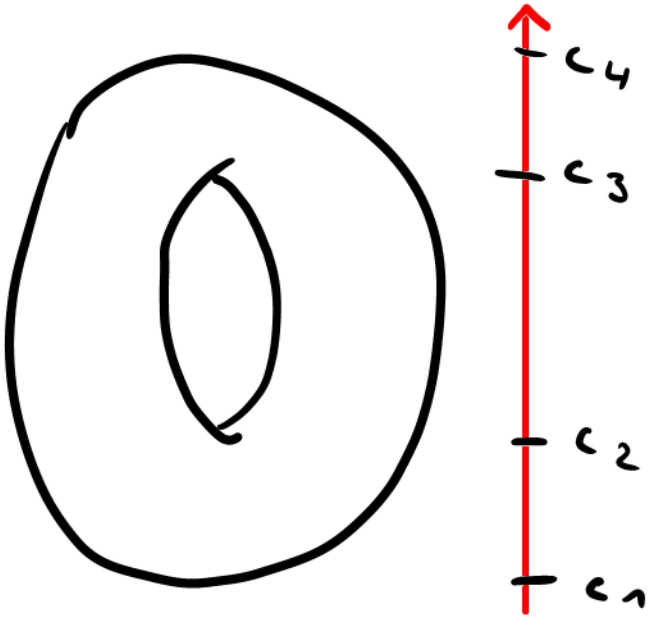
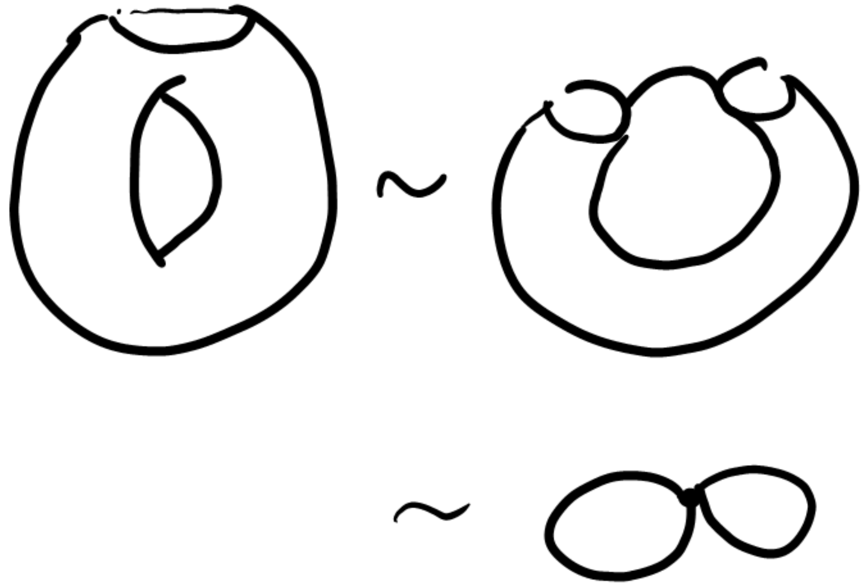


X_c , $c \in (c_3, c_4)$



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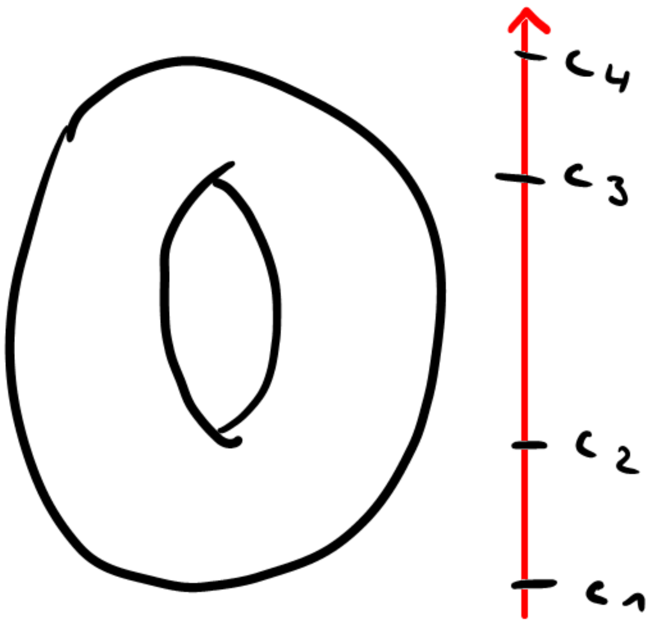
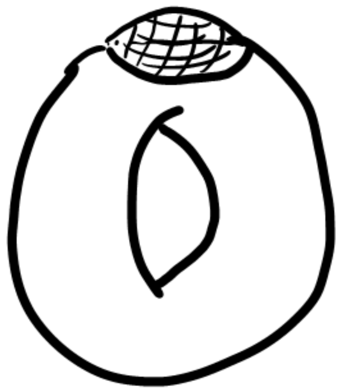
· f a height function

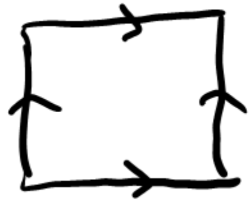


$X_c, \; c \in (c_4, +\infty)$

$\sim$



$\sim$



I) Classical Morse theory.

Assumption: X is a manifold,
$f_{|X} : X \longrightarrow \mathbb{R}$ smooth.

- $x \in X$ is a critical point when $T_x f = 0$.
- $c = f(x)$ is a critical value.

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- $x \in X$ is a critical point when $T_x f = 0$.
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Theorem: If $[a,b]$ contains no critical value, X_a is a deformation retract of X_b.

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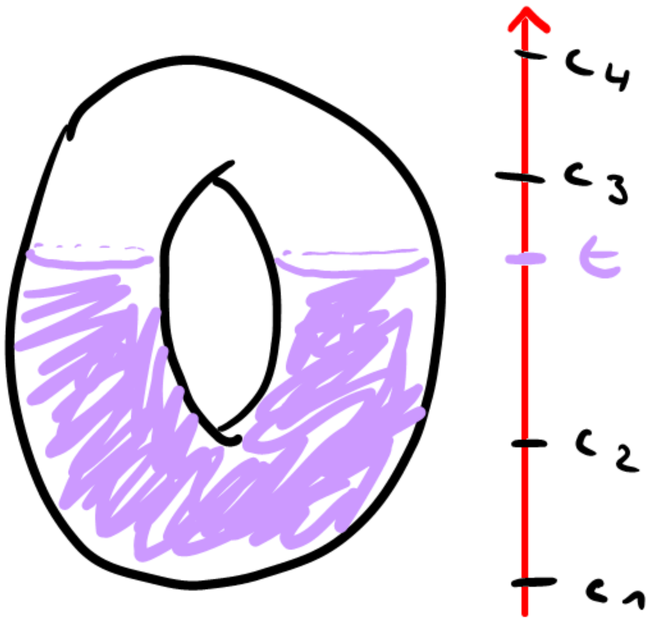
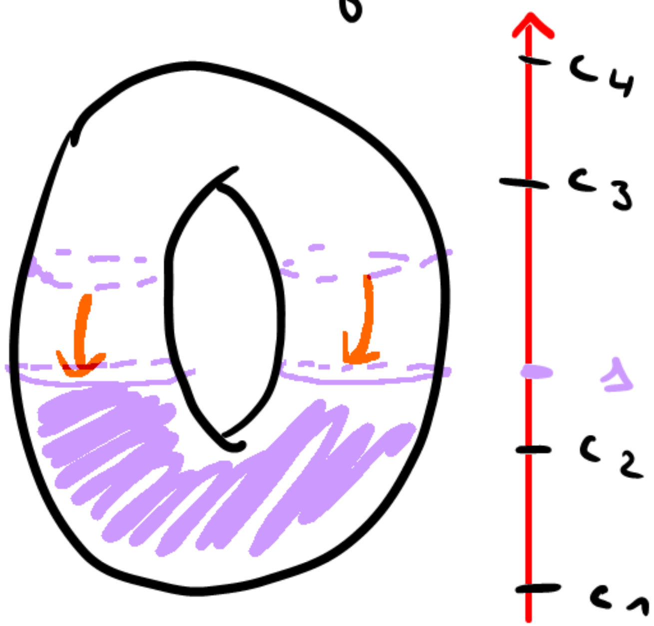
Proof idea:

Follow the flow of $-\frac{\nabla f}{\|\nabla f\|}$.





I) Classical Morse theory.

Assumption: X is a manifold,

$f|_X : X \longrightarrow \mathbb{R}$ smooth. (and proper!)

Theorem: If $x \in X$ is the sole critical point in $f^{-1}(c)$, and the Hessian of f at x is non degenerate,

$$X_{c+\varepsilon} \sim X_{c-\varepsilon} \cup_{\partial} \text{cell}$$

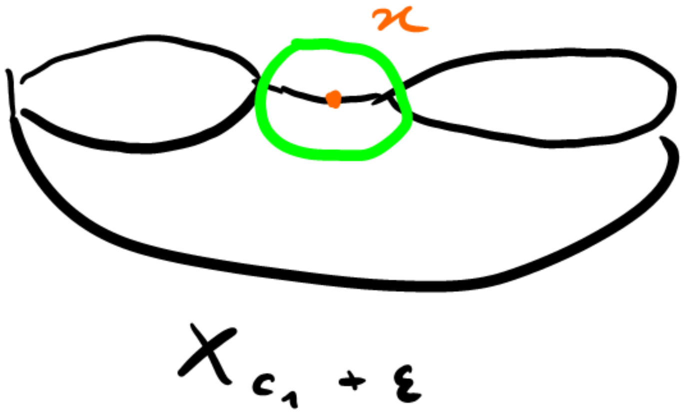
i.e

$X_{c+\varepsilon}$ is homotopic to $X_{c-\varepsilon}$ with a cell glued around x, of dimension determined by index of $Hf(x)$

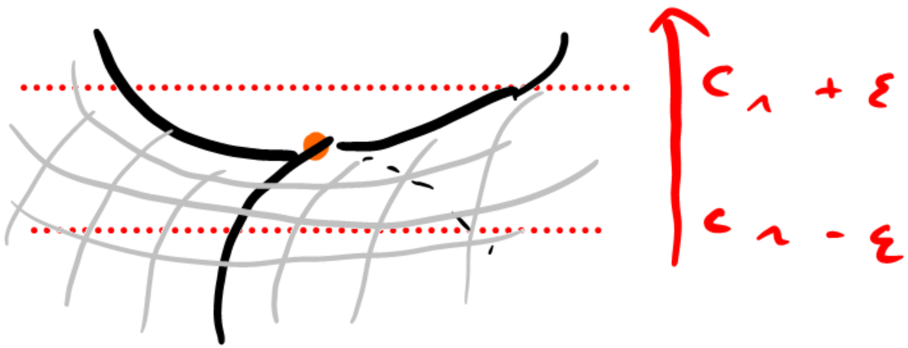
I) Classical Morse theory.

Assumption: X is a manifold,
$f|_X : X \longrightarrow \mathbb{R}$ smooth (and proper!)

Proof idea: • Apply flow of $-\frac{\nabla f}{\|\nabla f\|}$ outside of x;
• Around x, f looks like $\sum x_i^2 - \sum x_j^2$ in local coordinates.



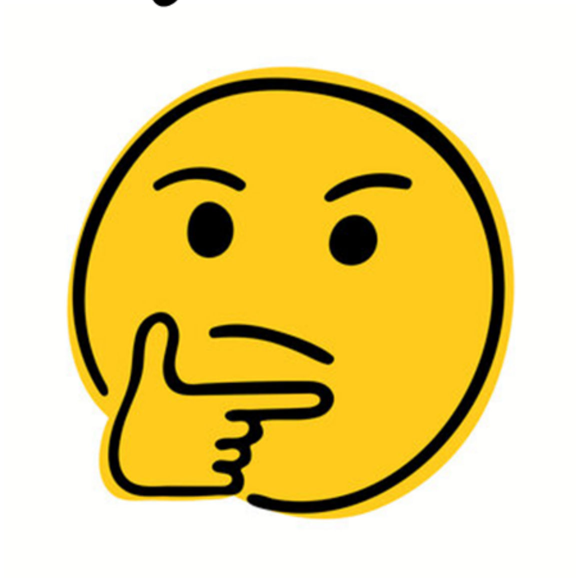
zoom



II) Morse theory for sets with positive reach

- Focusing on $X \subset \mathbb{R}^d$.

What can we say when X is NOT a submanifold ?



II) Morse theory for sets with positive reach

- Deep stuff on Morse theory for stratified sets (1988, Goresky, McPherson)

- Joseph Fu's idea: compare X to close submanifolds when possible!

II) Morse theory for sets with positive reach

Def: (Federer, 1959) Let $X \subset \mathbb{R}^d$.

$$\mathrm{reach}(X) = \sup\{ t \in \mathbb{R}^+ \mid d_X(x) \leq t \Rightarrow x \text{ has } 1 \text{ closest point in } X \}.$$

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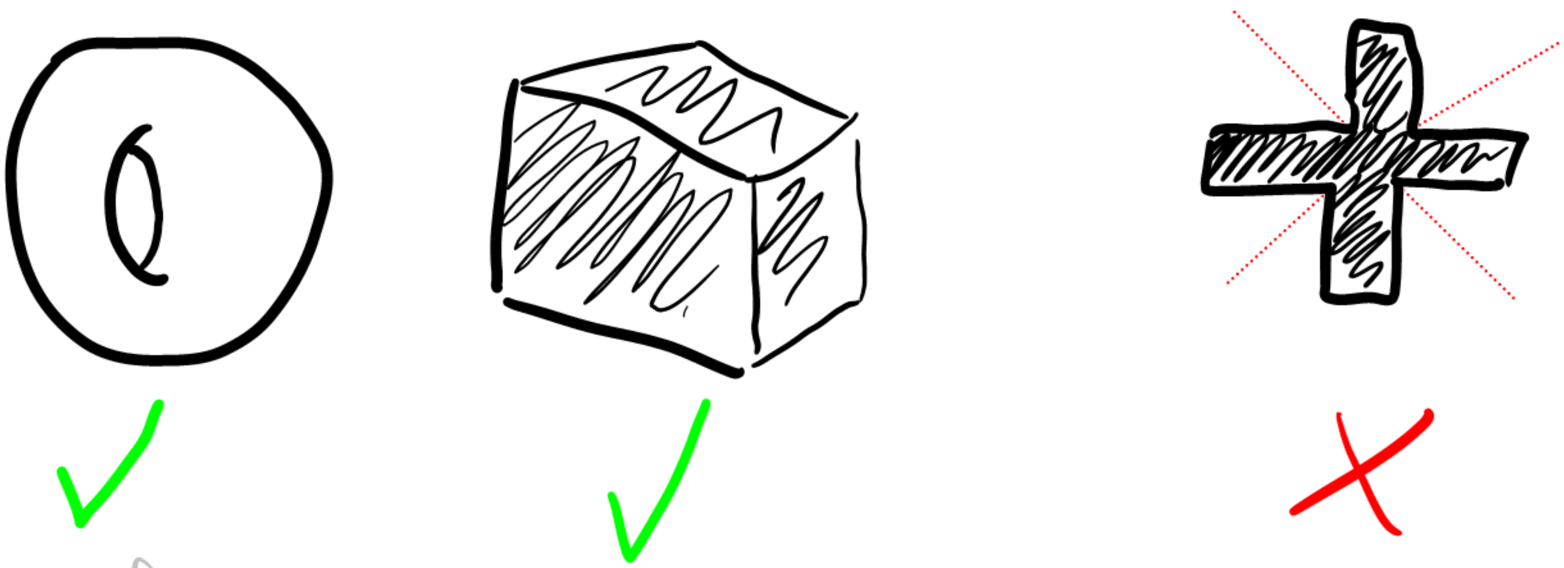


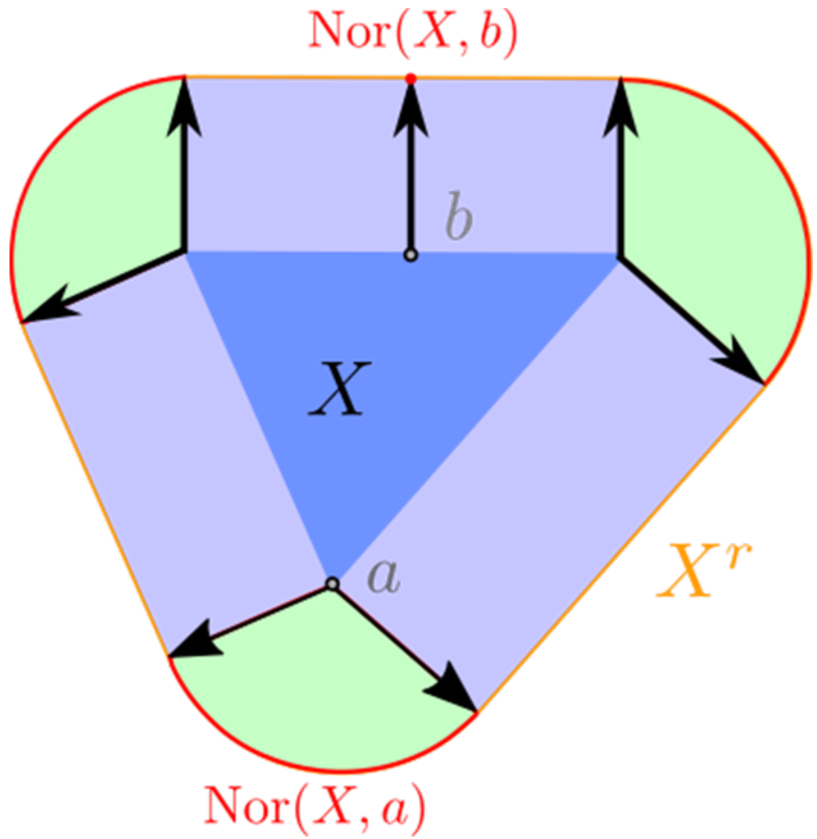
smooth   convex corners   concave corners

II) Morse theory for sets with positive reach

Def: · Let $x \in X$ with $\mathrm{reach}(X) > 0$.

$\mathrm{Nor}(X, x)$ cone of direction projecting back to X.

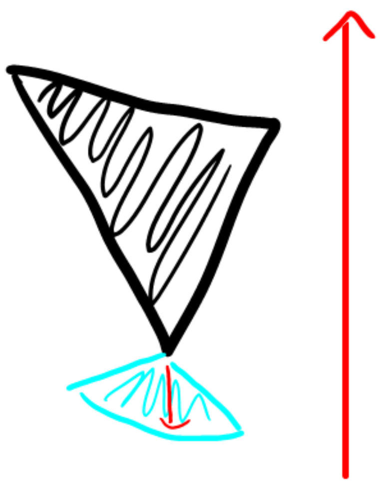
Example in $\mathbb{R}^2$:



II) Morse theory for sets with positive reach

- $x \in X$ with $\mathrm{reach}(X) > 0$ is critical when $-\nabla f(x) \in \mathrm{Nor}(X, x)$

Fu's Result (1989): When X has positive reach, the two Morse theorems stand



II) Morse theory for sets with positive reach

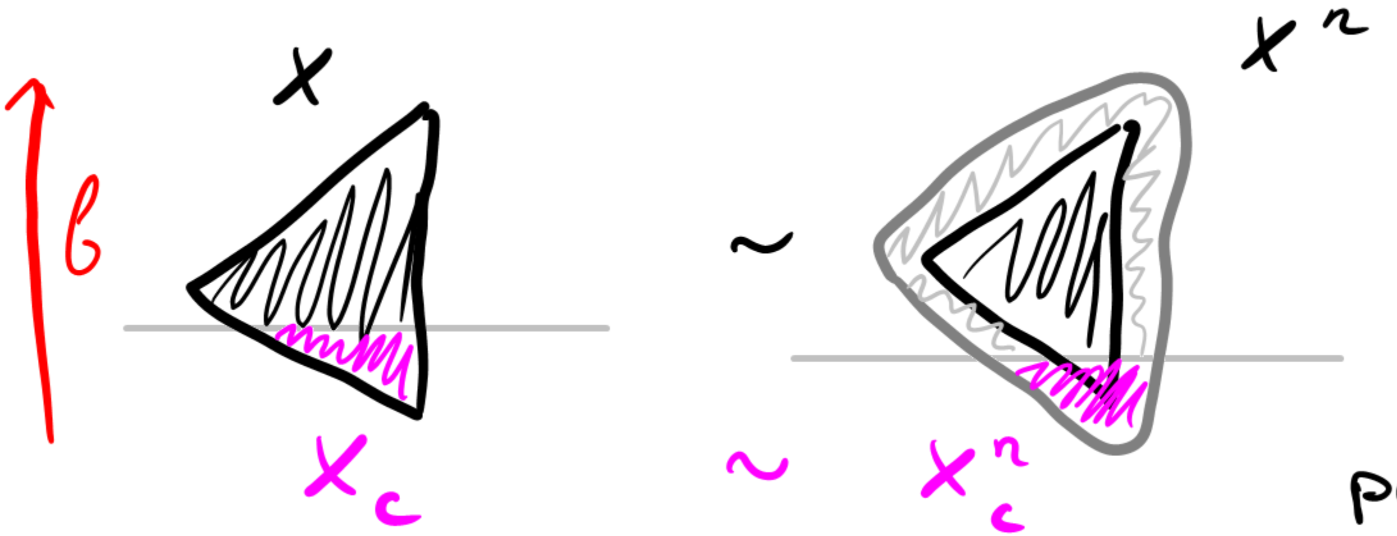
Proof Idea: Let $X^r = \{x \mid d_X(x) \le r\}$.

Then $X^r_c = X^r \cap f_r^{-1}(-\infty, c)$

is homotope to X_c when

c is a regular value and r small enough

and f_r is close to f.



Idea: Building homotopy using closest point projection.

II) Morse theory for sets with positive reach

Around a critical value c:

$$\begin{tikzcd}
X_{c+\varepsilon} \arrow[r, phantom, "\sim"] & X^r_{c+\varepsilon} \arrow[r, phantom, "\sim"] & X^r_{c-\varepsilon} \cup_\partial \text{cell} \\
X_{c-\varepsilon} \arrow[u] \arrow[r, phantom, "\sim"] & X^r_{c-\varepsilon} \arrow[u] &
\end{tikzcd}$$

Lemma: When r is small enough, the dimension of the cell added depend only on the curvature of X and the hessian of f.

III) Morse theory for Tubular Neighborhoods

<u>Idea:</u> Adapt Fu's work when reach$(X)=0$, with weaker assumptions.

<u>Def</u>: Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be locally Lipschitz.

Its Clarke Gradient at x is

$$\partial^* \phi(x) = \mathrm{Conv}(\{\lim \nabla\phi(x_i) \mid x_i \to x \ \& \ \nabla\phi(x_i) \text{ is defined}\})$$

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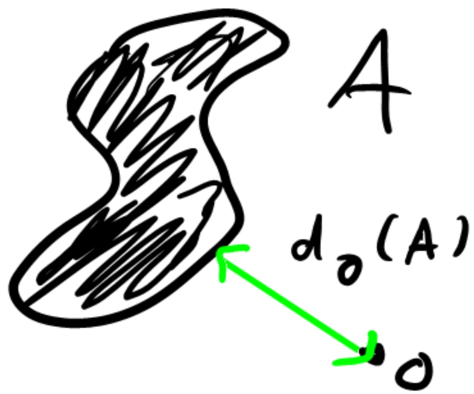
- $d_0(A) = \inf \{\|a\|,\ a \in A\}$



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- $d_0(A) = \inf\{\|a\|,\ a \in A\}$

- $\mathrm{reach}_\varepsilon(X) = \sup\{t \in \mathbb{R}^+ \mid d_X(x) \le t \Rightarrow d_0(\partial^* d_X(x)) \ge \varepsilon\}$

$(\mathrm{reach}_1(X) = \mathrm{reach}(X))$

III) Morse theory for Tubular Neighborhoods

• **Motivation:** If $d_0(\partial^* \phi(x)) \geq \epsilon$ on $\phi^{-1}(a,b]$ then $\phi^{-1}(-\infty,a]$ is a deformation retract of $\phi^{-1}(-\infty,b]$

$\Rightarrow$ Replace $\mathrm{reach}(X) > 0$ by $\mathrm{reach}_\epsilon(X) > 0$?

III) Morse theory for Tubular Neighborhoods

• **Motivation:** If $d_0(\partial^* \phi(x)) \geq \varepsilon$ on $\phi^{-1}(a,b]$ then $\phi^{-1}(-\infty,a]$ is a deformation retract of $\phi^{-1}(-\infty,b]$

$\Longrightarrow$ Replace $\mathrm{reach}(X) > 0$ by $\mathrm{reach}_\varepsilon(X) > 0$?

$\Rightarrow$ X^n is not smooth anymore

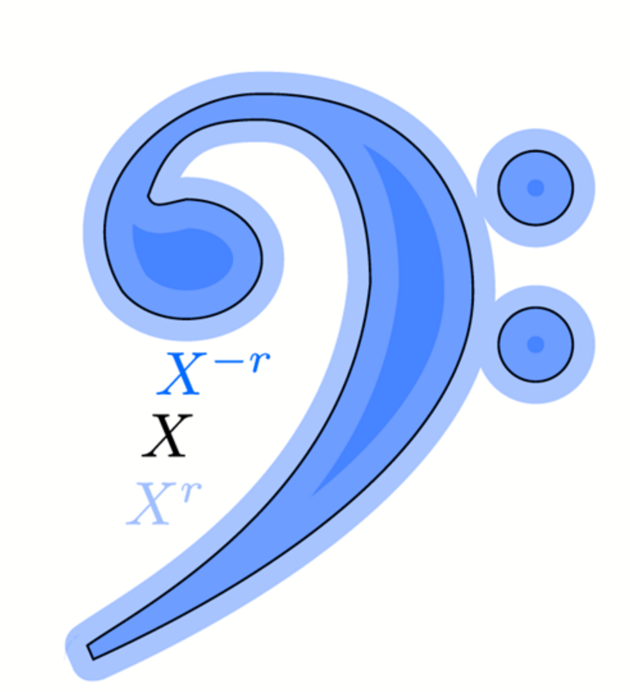
$\Rightarrow$ what is a critical point ?

III) Morse theory for Tubular Neighborhoods

Solution: Put $\neg X = \overline{\mathbb{R}^d \setminus X}$

and impose $\begin{cases} \partial X = \partial \neg X \\ \text{reach}(\neg X) > 0 \end{cases}$

and compare X to $X^{-r} = \{x \in \mathbb{R}^d \mid d_{\neg X}(x) \geq r\}$



III) Morse theory for Tubular Neighborhoods

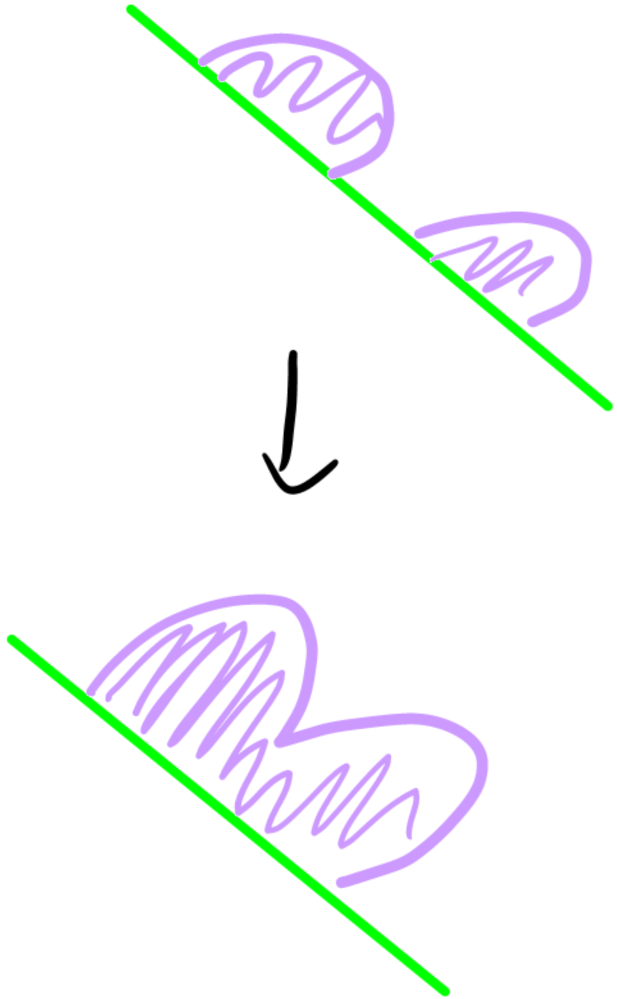
Theorem (AC): the class of sets verifying these assumptions is exactly the class of sets of the form

$$X = Y^{\varepsilon}$$

where $\inf \{ d_0(\partial^* d_Y(x)) \mid x \in \partial X \} > 0.$

III) Morse theory for Tubular Neighborhoods

<u>Def</u>: x is critical for $f_{|X}$ when $\nabla f \in \mathrm{Nor}({}^{\urcorner}X, x)$



III) Morse theory for Tubular Neighborhoods

Def: x is critical for $f_{|X}$ when $\nabla f \in \mathrm{Nor}(X,x)$

Theorem: (AC) · When c is a regular value,

$$X_c^{-r} \sim X_c$$

when r is small enough

· When c is a critical value,
the change in topology between $X^{-r}_{c+\varepsilon}$ and $X^{-r}_{c-\varepsilon}$
only depend on the curvatures of X and f at x
when r, ε are small enough.

Thank You!