Persistent homology and the continuity of generalized curvatures.

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Joint work with David Cohen-Steiner.

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Persistent day

General (vague) question:

If $\boxed{(d^*)\ X_n \longrightarrow X}$ with respect to a metric d^*,

"Simple sets"

- $\mathrm{Conv}(X_n) \longrightarrow \mathrm{Conv}(X)$?

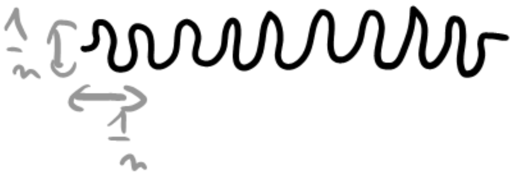
- Does this stand when X is not a "simple set"?

Choice of a metric

• (d_H) $X_n \longrightarrow X$ $\not\Rightarrow$ "$\mathrm{Curv}(X_n) \longrightarrow \mathrm{Curv}(X)$"

Hausdorff distance

Example



X_n

X

X_n are too curved!

Choice of a metric

• (d_H) $X_n \longrightarrow X \;\not\Rightarrow\;$ "$\mathrm{Curv}(X_n) \longrightarrow \mathrm{Curv}(X)$"

Hausdorff distance

• (d_N) $X_n \longrightarrow X \;\Longrightarrow\;$ "$\mathrm{Curv}(X_n) \longrightarrow \mathrm{Curv}(X)$"

✓

... by definition!

C^1-distance on Gauss map between hypersurfaces

Theorem: Continuity of curvatures holds for the metrics d_{Haus} and d_{Acy} on compact $\subset \mathbb{R}^d$.

Theorem: Continuity of curvatures holds for the metrics d_{Hom} and d_{Acy} on compact $\subset \mathbb{R}^d$.

"ε-homotopy metric" d_{Hom}:

$d_{Hom}(X,Y) < \varepsilon \iff$ $f : X \longrightarrow Y$, $g : Y \longrightarrow X$,
$H_Y : f \circ g \sim Id_Y$, $H_X : g \circ f \sim Id_X$

such that $\forall x \in X, \forall y \in Y, \forall t \in [0,1]$,

$$\begin{cases} \|f(x) - x\|, \|g(y) - y\| \leq \varepsilon \\ \|H_X(t,x) - x\|, \|H_Y(t,y)\| \leq \varepsilon \end{cases}$$

Theorem: Continuity of curvatures holds for the metrics d_{Hom} and d_{Acy} on compact $\subset \mathbb{R}^d$.

Acyclic metric d_{Acy}

$d_{Acy}(X,Y) \leq \varepsilon \iff$

$\exists\, \mathcal{R} \subset X \times Y,$

- $\forall (x,y) \in \mathcal{R},\ \|x-y\| \leq \varepsilon.$

&

$\forall x \in X,\ \mathcal{R}^Y_x = \{y \in Y,\ x\mathcal{R}y\}$
$\forall y \in Y,\ \mathcal{R}^X_y = \{x \in X,\ x\mathcal{R}y\}$

have the homology groups of a point.

Convergence in d_{Hom}, d_{Acy}

- Relaxation of the Fréchet distance

$$d_{Fré}(X,Y) = \inf \left\{ \varepsilon \text{ tq } \exists f : X \to Y, \text{ homeo}, \ \|f(x) - x\| \leq \varepsilon \right\}$$

Convergence in d_{Hom}, d_{Acy}

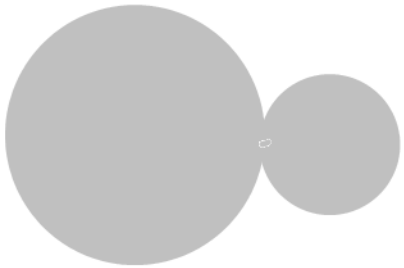
- <u>Example of convergence:</u>

If 0 is a weakly regular value of $f : \mathbb{R}^d \to \mathbb{R}$, then (d_{Acy}, d_{Hom})

$$f^{-1}(-\infty, t] \xrightarrow[t \to 0^+]{} f^{-1}(-\infty, 0]$$

by constructing a retraction using the flow of f.

<u>Ex:</u>



$f = d_X$

$$X^t = \{x, d_X(x) \le t\} \xrightarrow[(d_{Hom}, d_{Acy})]{} X$$

Two questions

- This is persistent day, where is the persistence?

- What do we mean by "Generalized curvatures"?

In particular, in what sense

$$\mathrm{Curv}(X_n) \xrightarrow[n \to +\infty]{} \mathrm{Curv}(X) \,?$$

Persistent Homology

Persistent homology : definition

A persistence module is a collection of vector spaces and linear maps.

$r \leq s \leq t$

$$\begin{tikzcd} \cdots \arrow[r, dashed] & M_r \arrow[r] \arrow[rr, bend left] & M_s \arrow[r] & M_t \arrow[r, dashed] & \cdots \end{tikzcd}$$

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<u>Example</u>: If $(X_t)_{t \in \mathbb{R}}$ is a filtration,

$$\cdots \dashrightarrow H_i(X_r) \longrightarrow H_i(X_s) \longrightarrow H_i(X_t) \dashrightarrow \cdots$$

We speak of persistent homology modules.

Persistent homology : decomposition

Under mild regularity conditions,
a persistence module can be decomposed
as a sum of interval modules $\mathbb{1}_I$

$$\cdots \dashrightarrow 0 \xrightarrow{0} \underbrace{\mathbb{K} \xrightarrow{\mathrm{id}} \mathbb{K}}_{I} \xrightarrow{0} 0 \dashrightarrow \cdots$$

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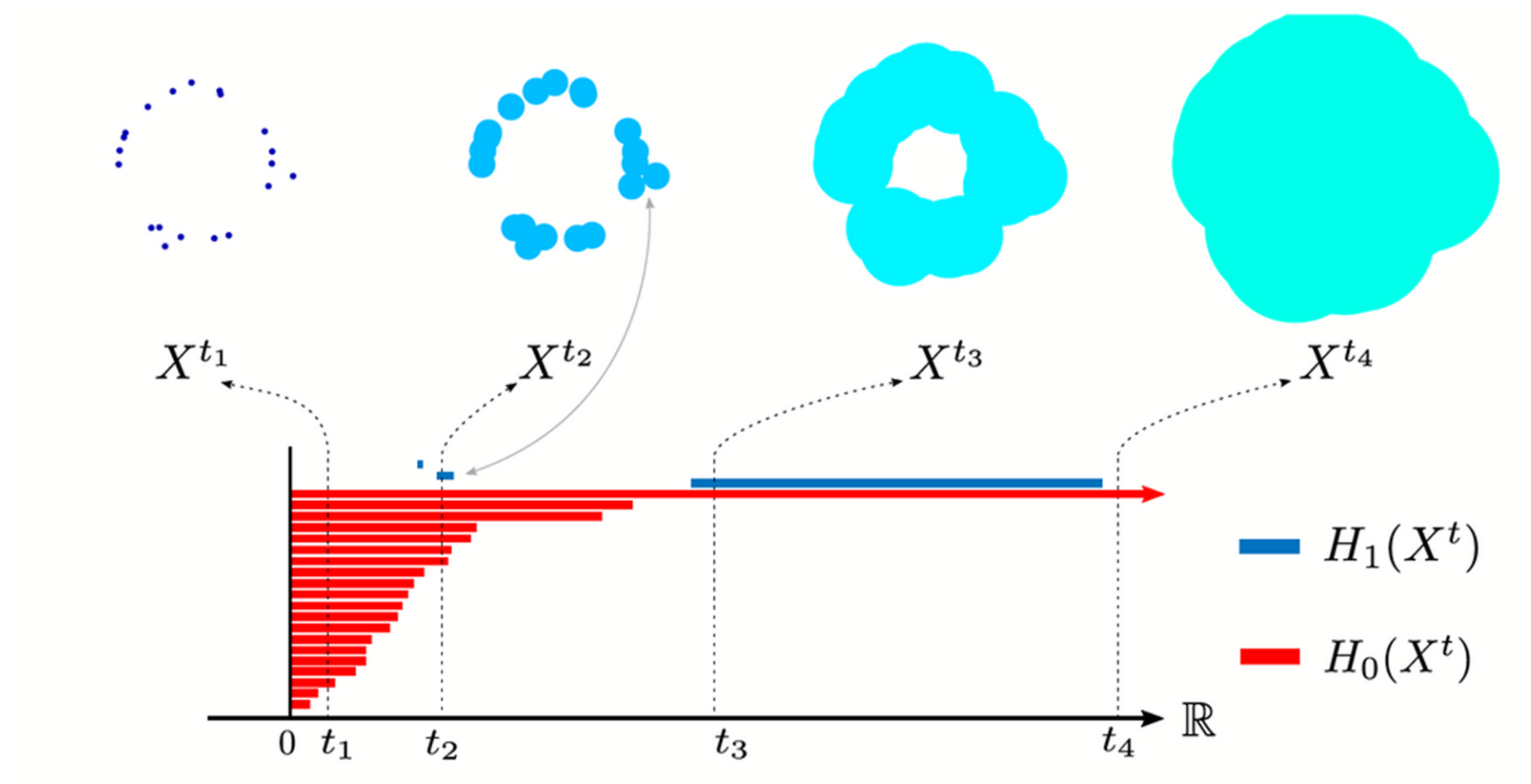
For persistent homology modules, each
interval's bounds correspond to the birth and death
of topological features.

Persistent homology : decomposition

Example: the growing offset filtration



Persistent homology: stability

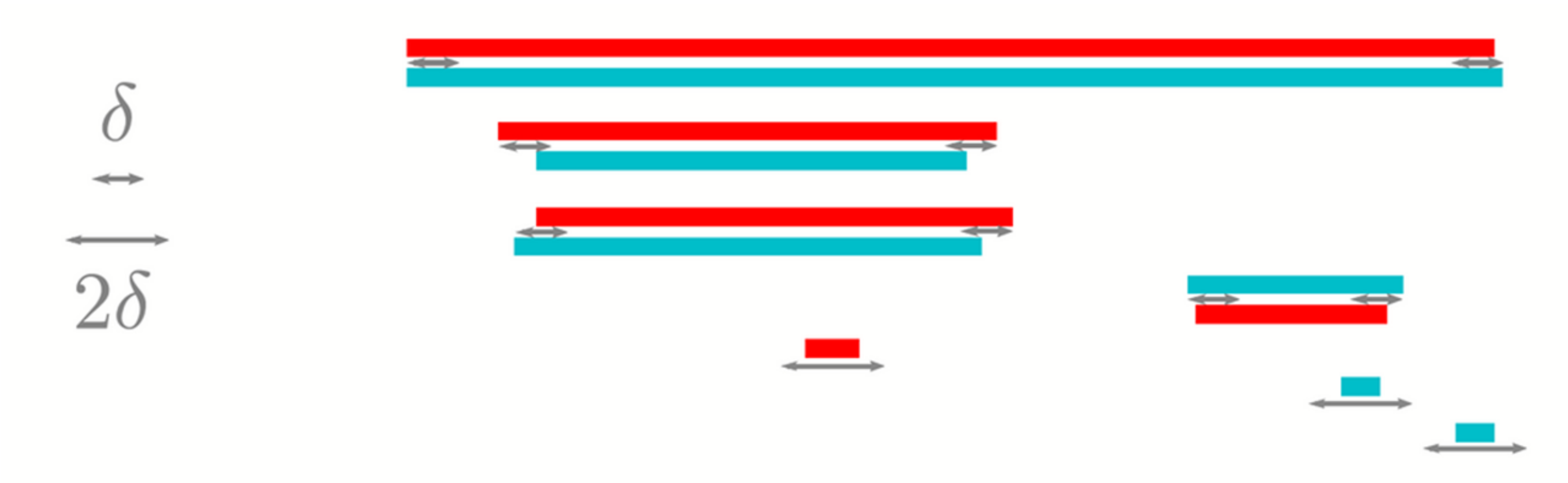
Persistence modules form a pseudo-metric space when equipped with the interleaving distance.

$d_I(M,N) :=$ infimum of δ such that

$$
\begin{tikzcd}
\cdots \arrow[r, dashed] & M_t \arrow[r] \arrow[dr] & M_{t+\delta} \arrow[r] \arrow[dr] & M_{t+2\delta} \arrow[r, dashed] & \cdots \\
\cdots \arrow[r, dashed] & N_t \arrow[r] \arrow[ur] & N_{t+\delta} \arrow[r] \arrow[ur] & N_{t+2\delta} \arrow[r, dashed] & \cdots
\end{tikzcd}
$$

Persistent homology: stability

Persistence diagrams are equipped with the bottleneck distance d_B defined as the infimum of δ-matchings.



Partial bijection moving bounds by less than δ.

Persistent Homology

If $(X_t)_{t \in \mathbb{R}}$ is a filtration, with well-defined persistent homology modules,

$$\chi(X_t) = \text{alternating sum of number of intervals containing } t.$$

Persistent Homology

If $(X_t)_{t\in\mathbb{R}}$ is a filtration, with well-defined persistent homology modules,

χ-Averaging Lemma:

If (X_t), (Y_t) are two filtrations, and $c>0$ with $d_I(H_i(X_t), H_i(Y_t)) \leq c$ for all i,

$$\int_a^b |\chi(X_t) - \chi(Y_t)| \leq c \times \left(N((X_t)_{t\in\mathbb{R}}) + N((Y_t)_{t\in\mathbb{R}}) \right)$$

Total number of bars in the persistent homology modules.

Persistent homology

For any $v \in \mathbb{S}^{d-1}$, let $h_v : \begin{cases} \mathbb{R}^d \longrightarrow \mathbb{R} \\ x \longmapsto \langle x, v \rangle \end{cases}$

Let $M_i(v, X)$ be the persistent homology modules $H_i\left(h_v(-\infty, t] \cap X\right)_{t \in \mathbb{R}}$

Persistent homology

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Let $M_i(v, X)$ be the persistent homology modules $H_i(h_v(-\infty, t] \cap X)_{t \in \mathbb{R}}$

Fact: $d_I(M_i(v, X), M_i(v, Y))$
$$\leq d_{Ham}(X, Y), d_{Aug}(X, Y)$$

Fact: $d_I(M_i(v,X), M_i(v,Y)) \le d_{Hom}(X,Y), d_{Acy}(X,Y)$

Idea (for d_{Hom})

$$\exists \begin{cases} f : X \longrightarrow Y & f \circ g \sim Id_Y \\ g : Y \longrightarrow X, & g \circ f \sim Id_X \end{cases}$$

Use the homotopies restricted to $X_t^v = X \cap h_v(-\infty, t]$

$Y_t^v = X \cap h_v(-\infty, t]$

to bound interleaving distance.

Fact: $$d_I(M_i(v,X), M_i(v,Y)) \le d_{Hom}(X,Y), \, d_{Acy}(X,Y)$$

Idea:
$$X^v_t = X \cap h_v(-\infty, t]$$
$$Y^v_t = X \cap h_v(-\infty, t]$$

$$c = d_{Hom}(X,Y)$$

$$\begin{tikzcd}
 & Y^v_t \arrow[dr, "g"] & \\
X^v_{t-c} \arrow[ur, "f"] \arrow[rr, hook] & & X^v_{t+c}
\end{tikzcd}$$

Generalized Curvatures

Generalized curvatures

- A consistent framework to describe the curvature of some possibly non-smooth sets.

- Consist of curvature measures

$$C_0(X, \cdot), \ldots, C_{d-1}(X)$$

and the more general Normal cycle N_X

$$N_X : \underset{\Lambda^{d-1}(\mathbb{R}^d \times S^{d-1})}{\omega} \longmapsto N_X(\omega) \in \mathbb{R}$$

Generalized curvatures

- **Definition for a C^2 submanifold $X \subset \mathbb{R}^d$.**

Let $\mathrm{Nor}(X) = \bigcup_{x \in X} \{x\} \times \mathrm{Nor}(X, x)$ be its unit normal bundle.

<u>Fact</u>: $\mathrm{Nor}(X)$ is a $(d-1)$-submanifold of $\mathbb{R}^d \times \mathbb{S}^{d-1}$!

Generalized curvatures

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Fact: $\mathrm{Nor}(X)$ is a $(d-1)$-submanifold of $\mathbb{R}^d \times \mathbb{S}^{d-1}$!

and for every $(x,n) \in \mathrm{Nor}(X)$,

- $\exists\ b_1, \dots, b_{d-1}$ ortho. basis of $n^\perp$ (Principal directions)
- $\exists\ \kappa_1, \dots, \kappa_{d-1} \in \mathbb{R}$ (Principal curvatures)

such that $\left(\frac{1}{\sqrt{1+\kappa_i^2}} b_i, \frac{\kappa_i}{\sqrt{1+\kappa_i^2}} b_i \right)_{1 \le i \le d-1}$

is an ortho. basis of $\underbrace{T_{(x,n)} \mathrm{Nor}(X)}_{\text{Tangent space of } \mathrm{Nor}(X) \text{ at } (x,n)} \subset \mathbb{R}^d \times \mathbb{R}^d$.

Generalized curvatures

When X is a $(d-1)$-submanifold:

$$C_k(X, \mathcal{U}) = \int_{\mathcal{U} \cap X} (\text{Sym. polynomial of } d^\circ (d-k-1) \text{ in } \bar{\kappa}) \, d\omega_X(x)$$

Generalized curvatures

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Particular case:

$$C_0(X,\mathcal{U}) = \int_{\mathcal{U}\cap X} \prod_{i=1}^{d-1} \kappa_i \, d\omega_X(x)$$

Gauss-Bonnet: $C_0(X, X) = \chi(X)$

Generalized curvatures

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Particular case :

$$C_{d-1}(X, \mathcal{U}) = \int_{\mathcal{U} \cap X} 1 \, d\omega_X(x) = \mathcal{H}^{d-1}(X \cap \mathcal{U}).$$

Boundary area

$$C_{d-2}(X, \mathcal{U}) = \int_{\mathcal{U} \cap X} \left(\sum_{i=1}^{d-1} \kappa_i\right) d\omega_X(x)$$

Mean curvature.

Generalized curvatures

When X is a submanifold

$$C_k(X,U) = \int_{(U\times S^{d-1})\cap \mathrm{Nor}(X)} \prod_{i=1}^{d-1} \frac{1}{\sqrt{1+\kappa_i^2}} \left(\text{Sym. Pol in } \bar{\kappa}\right) \times dH^{d-1}(x,n)$$

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$$C_{d-1}(X,U) = \int_{(U\times S^{d-1})\cap \mathrm{Nor}(X)} \prod_{i=1}^{d-1} \frac{1}{\sqrt{1+\kappa_i^2}} \, dH^{d-1}(x,n) = \mathcal{H}^{d-1}(X\cap U)$$

change of variables

Generalized curvatures: Normal cycles

Normal cycle N_X uses the language of currents to describe curvatures of X is a general manner.

N_X is a $(d-1)$-current on $\mathbb{R}^d \times \mathbb{S}^{d-1}$:

Some current vocabulary:

- $\partial T : \omega \longmapsto T(d\omega)$ Boundary.

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- $F(T) = \sup \{ T(\omega),\ \|\omega\|_\infty, \|d\omega\|_\infty \leq 1 \}$

$$= \inf \{ M(A) + M(S),\ T = A + \partial S \}$$

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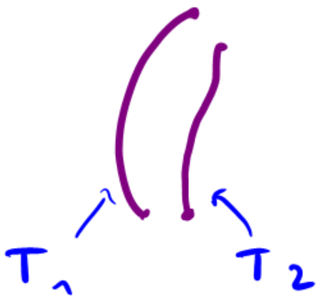
Ex: T_1, T_2 currents integrating over manifolds w/ boundary.



$F(T_1 - T_2) \le M(A) + M(S)$

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- Slice by $\pi : \mathbb{R}^d \times \mathbb{S}^{d-1} \longrightarrow \mathbb{S}^{d-1}$

$\langle T, \pi, v \rangle$ is a 0-current thought as T restricted to $\pi^{-1}(v) = \{ (x, n) \in \mathrm{Support}(T), n = v \}$

(d-1) current

Generalized curvatures : Normal cycles

$$\tilde{b}_i := \left(\frac{1}{\sqrt{1+\kappa_i^2}} b_i, \frac{\kappa_i}{\sqrt{1+\kappa_i^2}} b_i \right)$$

$$N_X(\omega) := \int_{Nor(X)} \omega_{(x,n)}(\tilde{b}_1, \dots, \tilde{b}_{d-1}) \, dH^{d-1}(x,n)$$

Functional integration over the unit normal bundle.

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Generalized curvatures : Normal cycles

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• $N_X(\alpha \wedge \omega') = 0$ $\quad \forall \omega'$ if $\alpha(x,n) : (u,v) \longmapsto u \cdot n$ is the contact form

N_X is "Legendrian"

Generalized curvatures: Normal cycles

<u>Fact</u>: • For almost all $v \in \mathbb{S}^{d-1}$, $h_{v|X} : x \mapsto \langle v, x \rangle$ is Morse

• $y \in X$ is critical for $h_{v|X} \Leftrightarrow (y, -v) \in \mathrm{Nor}(X)$

Generalized curvatures: Normal cycles

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• $y \in X$ is critical for $h_{v|X} \iff (y, -v) \in \mathrm{Nor}(X)$

• If $H_{v,t} := \{ x \in \mathbb{R}^d, \langle x, v \rangle \leq t \}$,

$$\chi(X \cap H_{v,t}) = \text{alternating sum of critical points of } h_{v|X} \text{ in } H_{v,t}$$

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$\Rightarrow$ Equivalently, $\underbrace{\langle N_X, \pi, -v \rangle}_{\text{0-current obtained as } "N_{X|\mathbb{R}^d \times \{-v\}}"} (\mathbb{1}_{H_{v,t}}) = \chi(X \cap H_{v,t})$

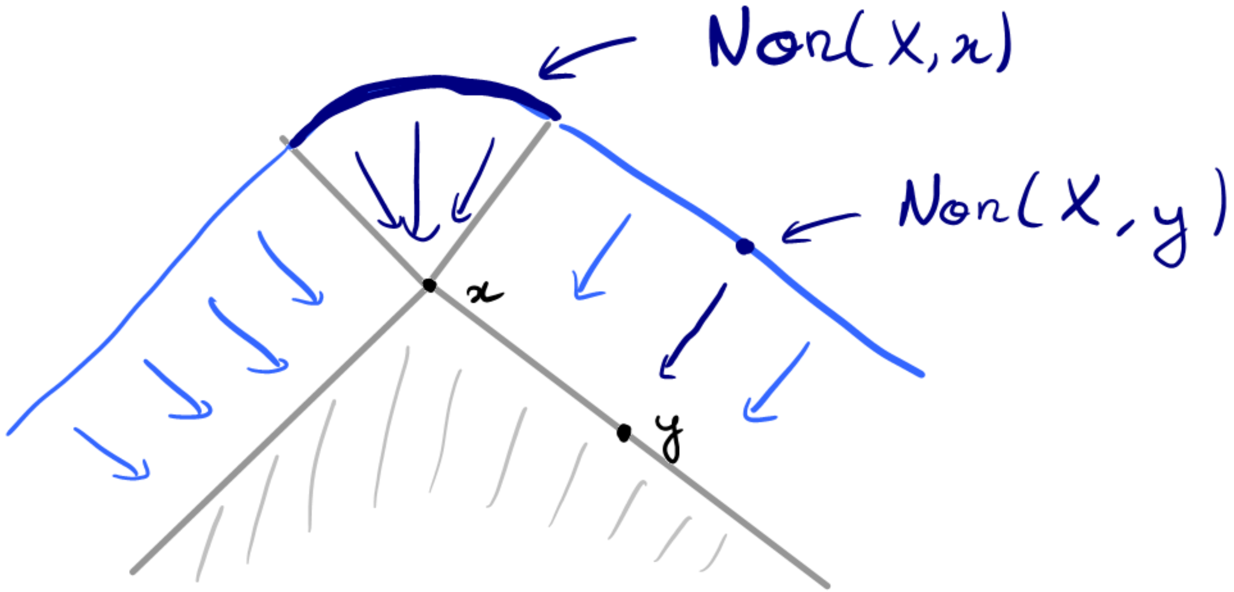
Generalized curvatures: Normal cycles

When X is convex, define $\boxed{Nor(X) = \bigcup_{x \in X} \{x\} \times Nor(X,x)}$

with $Nor(X,x)$

$=$

Set of direction projected back onto X



Generalized curvatures: Normal cycles

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Fact: In this case, $\mathrm{Nor}(X)$ is also a $(d-1)$-submanifold, and we define N_X the same way.

⚠ Principal curvatures can be infinite! (e.g at an edge)

Generalized curvatures: Normal cycles

Fu's uniqueness Theorem (1994).

For any $X \subset \mathbb{R}^d$, there is at most one integral current T such that

- $T(d\omega) = 0$ $\quad (\partial T = 0)$
- $T(\alpha \wedge \omega') = 0$ $\quad$ (Legendrian)
- For almost every $(\nu, t) \in \mathbb{S}^{d-1} \times \mathbb{R}$,

$$\langle T, \pi, -\nu \rangle (\mathbb{1}_{H_{\nu,t}}) = \chi(X \cap H_{\nu,t})$$

If such a T exists, we call it the normal cycle of X, denoted by N_X.

Generalized curvatures: Normal cycles

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Consequence: if 3 among $X, Y, X \cap Y, X \cup Y$ admit a normal cycle, then so does the 4th with $N_{X \cap Y} + N_{X \cup Y} = N_X + N_Y$

Generalized curvatures : Normal cycles

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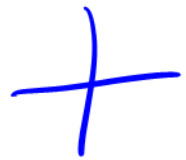
Ex : Normal cycle of a cross 

$$N_{+} = N_{|} + N_{-} - N_{\cdot}$$

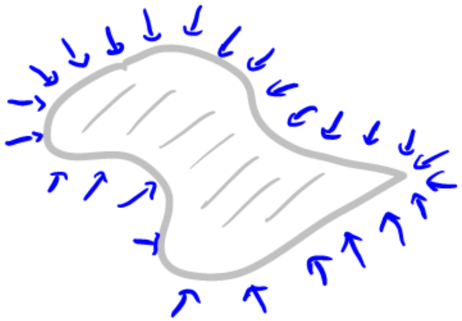
$\Rightarrow$ Sets with reach 0 have more complicated normal cycles than "integrating over the normal bundle".

Generalized curvatures : Normal cycles

History on Normal cycles

- Submanifolds, convex sets, more generally for sets with positive reach (83, Zähle)

Positive reach: there is a neighborhood of X on with there exist only one closest point in X.



Intuitively: smooth sets / domain with convex corners.

Generalized curvatures: Normal cycles

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- generic, locally finite union of sets with positive reach (86', Zähle & Rataj)

Generalized curvatures: Normal cycles

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Generalized curvatures: Normal cycles

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- Compact, definable sets in o-minimal structure (2004, Bernig) X

Generalized curvatures : Normal cycles

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- Compact, definable sets in o-minimal structure X (2004, Bernig)

- Sublevel sets of $h = f - g$ at a weakly regular value.
  convex.
  (2017, Pokorny & Rataj)

Theorem (A. C, Cohen-Steiner)

Assume (X_n) is a sequence of smooth compact sets with either

$$(d_{Haus})\quad X_n \longrightarrow X$$

$$(d_{Acy})\quad X_n \longrightarrow X,$$

and such that $\limsup_{n \to +\infty} M(N_{X_n}) < +\infty$.

Then X admits a normal cycle,

and $N_{X_n} \xrightarrow[n \to +\infty]{} N_X$

Sketch of proof.

Current compactness: extract so that $N_{X_n} \longrightarrow T$. (F)

Bound on number of critical points of bars in $\mathrm{dgm}(h_{v|X_n})$:

$$\limsup_{n \to +\infty} \int_{\mathbb{S}^{d-1}} N(v, X_n) \leq \limsup_{n\to+\infty} M(N_{X_n}) < +\infty$$

Total number of bars in the Persistence modules $H_i(X_n \cap h_v^{-1}(-\infty, t])_{t \in \mathbb{R}}$

$= M_i(v, X_n)$

Sketch of proof.

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Bound on number of critical points of bars in $dgm(h_{v|X_n})$:

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$M_i(v, X_n) \longrightarrow M_i(v, X)$ & lower semicontinuity of N:

$N(v, X) < +\infty$ for almost every $v \in S^{d-1}$.

So that

$f_n : (v, t) \longmapsto \chi(X_n \cap H_{v,t})$ converges in L^1

to $f : (v, t) \longmapsto \chi(X \cap H_{v,t})$

by the earlier χ-averaging lemma.

Further extract so that a.e in (v,t),

$$f_n(v,t) \xrightarrow[n \to +\infty]{} f(v,t)$$

so that

$$\langle T, \pi, \cdot v \rangle (\mathbb{1}_{H_{v,t}}) = \lim_{n \to +\infty} \langle W_{X_n}, \pi, \cdot v \rangle (\mathbb{1}_{H_{v,t}})$$

$$= \lim_{n \to +\infty} \chi(X_n \cap H_{v,t})$$

$$= \chi(X \cap H_{v,t})$$

so that the uniqueness theorem is satisfied.

Further extract so that a.e in (ν, t),

$$f_n(\nu,t) \xrightarrow[n \to +\infty]{} f(\nu,t)$$

so that

$$\langle T, \Pi, \cdot \nu \rangle (1\!\!1_{H_{\nu,t}}) = \lim_{n \to +\infty} \langle N_{X_n}, \Pi, \cdot\nu \rangle (1\!\!1_{H_{\nu,t}})$$

because $N_{X_n} \to T$

$$= \lim_{n \to +\infty} \chi(X_n \cap H_{\nu,t})$$

$$= \chi(X \cap H_{\nu,t})$$

Thanks to persistence.

so that the uniqueness theorem is satisfied.

$$\Downarrow$$

$$\boxed{T = N_X}$$

Application: normal cycle for compact set definable in a o-minimal structure.

Idea

if X is compact, definable, the generalized Łojasiewicz inequality holds

$$\boxed{\|\nabla_* d_X(x)\| \geq \varphi(d_X(x))}$$

for some definable function φ positive on $]0,\varepsilon]$ for some small $\varepsilon > 0$.

Application: normal cycle for compact set definable in a o-minimal structure.

- Consequence: One can build a retraction from $X^t = \{x \,, d_X(x) \le t\}$ to X, with finite length, so that

$$\boxed{(d_{A_{cy}}, d_{Hom}) \quad X^t \xrightarrow[t \to 0]{} X}$$

- Moreover, one can show

$$\boxed{\limsup_{\varepsilon \to 0} M(N_{X^\varepsilon}) < +\infty}$$

so that our theorem applies.

Conjecture (Fu)

Sets admitting a normal cycle are all of the form

$$X = \bigcap_{n \in \mathbb{N}} X_n$$

where X_n is non-increasing

and $\limsup_{n \to +\infty} M(N_{X_n}) < +\infty$

Thank you

for listening!