Intrinsic Persistent Volumes

Ongoing work with David Cohen-Steiner

Antoine Commaret

12/05/2022


WORK IN PROGRESS

The context

Say we know of K a compact set approximating an ideal object $\tilde{K}$ in $\mathbb{R}^d$. We want to know a geometric quantity $\varphi(\tilde{K})$.

How do we estimate $\varphi(\tilde{K})$ from K ?

The context

Offsets

Let $r \geq 0$. We put $A^r = \{x \in \mathbb{R}^d | d_A(x) \leq r\}$.

The Hausdorff distance

We put $d_H(\tilde{K}, K) = \inf\{r \geq 0, K \subset \tilde{K}^r, \tilde{K} \subset K^r\}$ the Hausdorff distance between $\tilde{K}$ and K.

Plug-In Method

Denoting $\varepsilon = d_H(\tilde{K}, K)$, we seek an estimator $\hat{\varphi}(K)$ such that

$$\left|\varphi(\tilde{K}) - \hat{\varphi}(K)\right| \leq f_{\varphi,\tilde{K}}(\varepsilon)$$

with $f_{\varphi,\tilde{K}}(s) \underset{s\to 0}{\longrightarrow} 0$ as fast as possible.

Simply take $\hat{\varphi}(K) = \varphi(K)$?

Plug-In Method

Comes out as **unsatisfactory** :

- Not always well defined.
- Could be faster. We aim for a **linear** rate of convergence with relaxed assumptions.

Plug-In Method

Comes out as **unsatisfactory** :

- Not always well defined.
- Could be faster. We aim for a **linear** rate of convergence without too general assumption.

-> We thus need a **consistent** and **non-parametric** definition of quantities akin to areas and perimeters.

Intrinsic Volumes

Reach of a compact set

Given $A \subset \mathbb{R}^d$, we put $\text{reach}(A)$ the distance to A under which every point has a **unique** closest point in A.

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Let $A \subset \mathbb{R}^d$ be a compact set with reach $r > 0$.

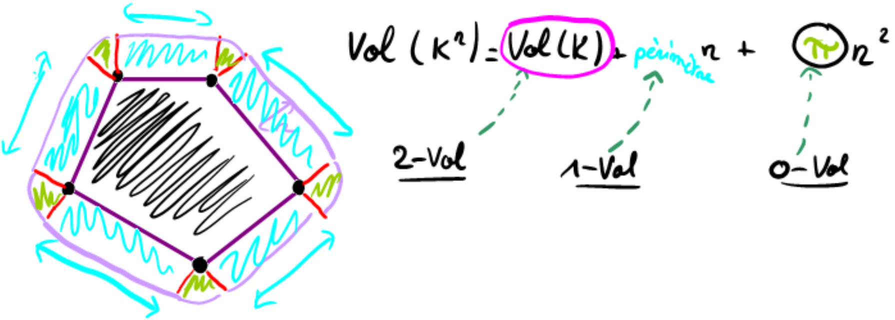
Steiner's Formula

Over $[0, r]$, $t \mapsto \mathrm{Vol}(A^t)$ is a polynomial whose coefficients define the **intrinsic volumes** of A.

$$\mathrm{Vol}(A^t) = \sum_{i=0}^{d} \omega_i \, V_{d-i}(A) t^i$$

Intrinsic Volumes : Example

Computing Steiner's Polynomial of a convex polygon in $\mathbb{R}^2$.



Intrinsic Volumes : Properties

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Figure 1 – A Happy Man

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- V_0 is in fact the **Euler Characteristic** χ (*Gauss-Bonnet Theorem*).
- $V_d(K)$ is the Lebesgue measure.
- $V_{d-1}(K)$ is the area.

Difficulties

Steiner's Formula **fails** when reach $= 0$

Could this condition be relaxed ?

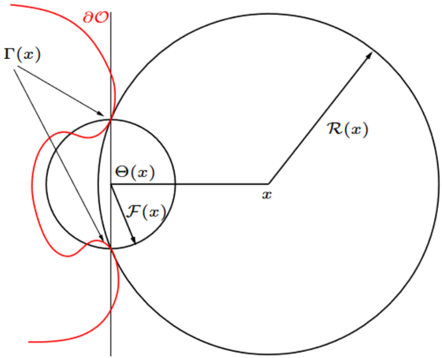
Solution we have come up with

- Considering set with ”positive $\mu -$ reach”, for $\mu \in]0, 1]$.
- Using a novel formula, the **Principal Kinematic Formula**.

About the μ-reach

Definition

$\text{reach}_\mu(A)$ is the distance to A under which, for any x, and given two closest points a, b to x on K, the cosinus of the half angle between $\vec{xa}$ and $\vec{xb}$ is bigger than μ.



Principal Kinematic Formula

Principal Kinematic Formula

Let A, B be compact sets of positive $\mu -$ reach for any $\mu \in]0,1]$. Denote G the group of affine isometries of $\mathbb{R}^d$ equiped with an invariant measure μ. Then,

$$\int_{g\in G} \chi(A \cap gB)\, \mathrm{d}\mu = \sum_{i=0}^{d} c_{d,i}\, V_i(A)\, V_{d-i}(B)$$

Using the Kinematic Formula

Rediscovering Steiner's Polynomial

Let $r \geq 0$. We arrive at Steiner's Polynomial through another road by taking $B = B_r$ a ball of radius r.

$$\int_{g \in G} \chi(A \cap gB_r)\,\mathrm{d}\mu = \sum_{i=0}^{d} \omega_i\, V_{d-i}(A) r^i$$

Persistence Steps in

χ is a topological quantity

Given a fixed g, $r \mapsto \chi(A \cap gB_r)$ is in fact the alternated sum of Betti Numbers of the persistence diagram associated to the function :

$$d^A_{g(0)} = \begin{cases} A & \to & \mathbb{R}^+ \\ x & \mapsto & \|x - g(0)\| \end{cases}$$

Persistence steps in

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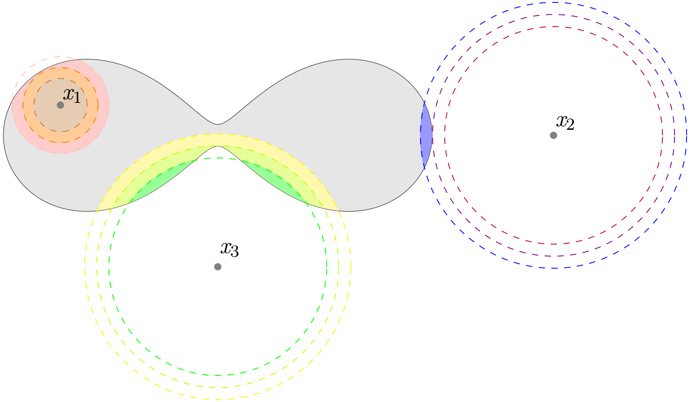

Persistence Steps In

Key Idea

We have

$$\sum_{i=0}^{d} \omega_i \, V_{d-i}(A) r^i = \int_{\mathbb{R}^d} \chi(\mathrm{diag}_y^A)\, \mathrm{d}y = Q_A(r)$$

and thus it **suffices to uniformly approximate the diagram of** d_y^A : our surrogate for Steiner's polynomial will be

$$\hat{Q}_A(r) = \int_{y \in \mathbb{R}^d} \chi(\widehat{\mathrm{diag}}_y^A)\, \mathrm{d}y$$

given a diagram approximate $\widehat{\mathrm{diag}}_y^A$.

Let's not forget our aim

How do we approximate the diagram ?

Image Persistence

Let's go back to $\tilde{K}$ and K. Denote $H^s_\bullet$ the persistence module induced by the function $d_y^{K^s}$ applying any homology functor, meaning $H^s_a = H(K^s \cap B(y,a))$.

We obtain the ***image persistence module***, denoted $\widehat{\operatorname{diag}}_y^K$:

$$
\begin{array}{ccccccc}
\cdots\rightarrow & H_a^{3\varepsilon} & \longrightarrow & H_b^{3\varepsilon} & \cdots\rightarrow \\
 & \uparrow & & \uparrow & \\
\cdots\rightarrow & H_a^{\varepsilon} & \longrightarrow & H_b^{\varepsilon} & \cdots\rightarrow
\end{array}
$$

Noisy Domains Theorem

Taking inspiration from a previous article, we have the following result :

Noisy Domains Theorem

$$d_B(\mathrm{diag}_y^{\tilde{K}^{2\varepsilon}}, \widehat{\mathrm{diag}}_y^{K}) \leq \frac{4\varepsilon}{\mu}$$

L1 Approximation of Steiner's Polynomial

L1 Linear Approximation

For any $R > 0$, we have

$$\| Q_{\tilde{K}^{2\varepsilon}} - \hat{Q}_K \|_{1,[0,R]} \leq \frac{4\varepsilon}{\mu} \int_{y \in \mathbb{R}^d} G_y^R \, \mathrm{d}y$$

Computing back the coefficient

Computing the coefficients of $\hat{Q}_K$ through a linear projection ensures that we have an estimation **converging at a linear rate :**

$$\left| V_i(\tilde{K}^{2\varepsilon}) - \hat{V}_i(K) \right| \leq O_{i,\tilde{K}}\left(\frac{\varepsilon}{\mu}\right)$$

Conclusion and questions

- Does not require any regularity on K : could be voxels, point clouds.
- Is there a way to optimize the computation over the parameter R ?
- How close is $V_i(\tilde{K}^{2\varepsilon})$ to $V_i(\tilde{K})$?

Remerciements

Thank you for listening !