Generalized Morse Theory for certain subsets of $\mathbb{R}^d$

Antoine Commaret

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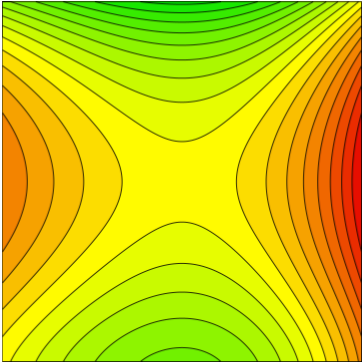
Sublevel sets topology

Let $X \subset \mathbb{R}^d$

Let $f : \mathbb{R}^d \to \mathbb{R}$



Sublevel sets topology

Evolution of $c \mapsto X_c = X \cap f^{-1}(]-\infty, c])$

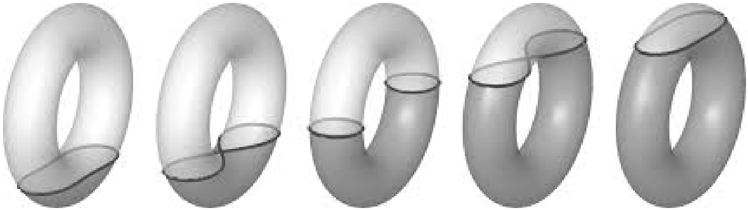
Classical Morse Theory

Smooth object in $\mathbb{R}^d$

Assume $X \subset \mathbb{R}^d$ and f are smooth.

- $x \in X$ is a **critical point** when $\nabla f(x)$ is orthogonal to X at x.
- At a critical point, define the "Hessian of $f_{|X}$ at x" as a linear combination of the **Hessian** of f at x and the **second fundamental form** of X at x.

Torus and a height function



Morse Theory Theorem - Isotopy Lemma

Assume $[a, b]$ does not contain any critical value.

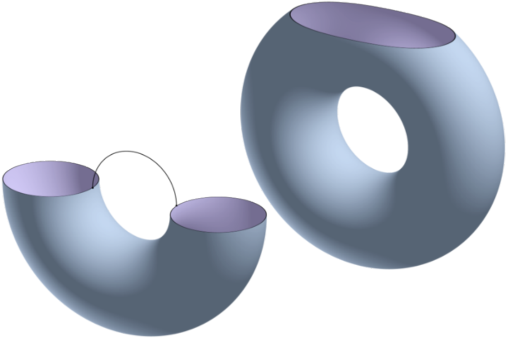
Then X_a **is a deformation retract of** X_b

Morse Theory Theorem - Handle Attachment lemma

Suppose that for all $\varepsilon > 0$ small enough, $f^{-1}([c-\varepsilon, c+\varepsilon])$ contains only one non-degenerate critical point of index λ.

Then $X_{c+\varepsilon}$ **has the homotopy type of** $X_{c-\varepsilon}$ with a λ-cell attached.

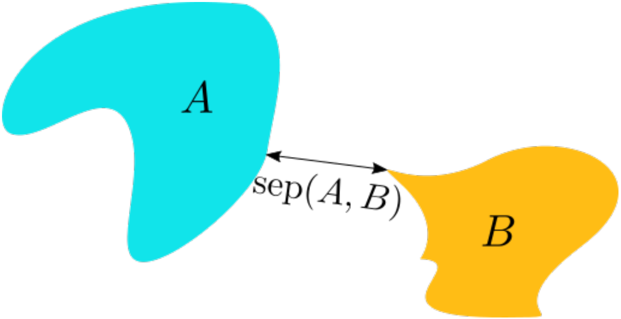
Morse Theory Theorem - Handle Attachment lemma



Vocabulary

Separation distance

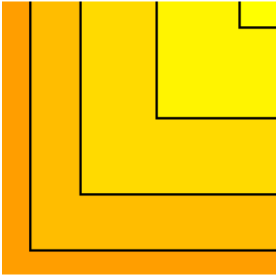
Define $\operatorname{sep}(A, B) = \inf_{(a,b)\in A\times B} \|a - b\|$.



Clarke Gradient

Definition.

If $\phi : \mathbb{R}^d \to \mathbb{R}$ is locally lip. its **Clarke Gradient** $\partial^* \phi(x)$ at x is the convex hull of the sets of limits $\lim_{h_i \to 0} \nabla \phi(x + h_i)$.



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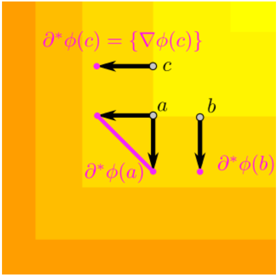

Critical points

Idea : Critical Points x **of a lip function** ϕ **are such that** $0 \in \partial^* \phi(x)$

Approximate Flow Lemma

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Approximate Flow Lemma.

Let $a < b$, ϕ locally lipschitz. Assume

$$\inf_{x \in \phi^{-1}(]a,b])} \operatorname{sep}(\partial^* \phi(x), \{0\}) > 0$$

Then $\phi^{-1}(]-\infty, a])$ **is a deformation retract of** $\phi^{-1}(]-\infty, b])$.

Distance to a closed set in $\mathbb{R}^d$

$$d_X(x) = \inf_{y \in X} \|x - y\|$$

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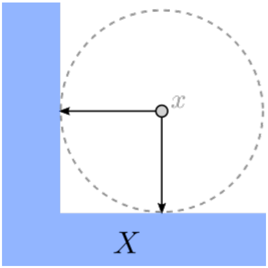
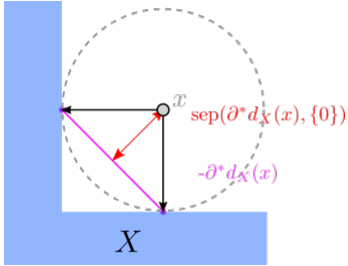


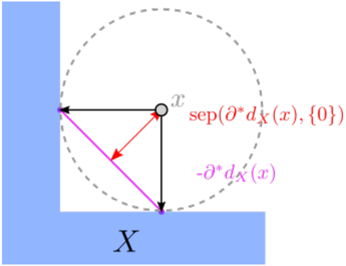
Clarke gradient of a distance function

$\partial^* d_X(x)$ has a geometrical meaning



Clarke gradient of a distance function

$\mathrm{sep}(\partial^* d_X(x), \{0\})$ measures how "flat" the angles between two closest points of x in X are at worst.

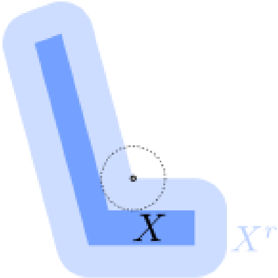


μ-reach

Definition

$\text{reach}_\mu(X) = \sup\{t \in \mathbb{R}, d_X(x) \leq t \implies \text{sep}(\partial^* d_X(x), \{0\}) \geq \mu\}$

- $\text{reach}_\mu(X) > 0$ means that there is a neighborhood of X in which **the angle between two directions of closest points from x in X cannot be "too flat."**.

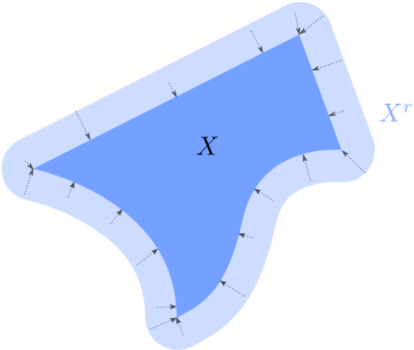


reach

Definition

$\text{reach}_\mu(X) = \sup\{t \in \mathbb{R}, d_X(x) \leq t \implies \text{sep}(\partial^* d_X(x), \{0\}) \geq \mu\}$

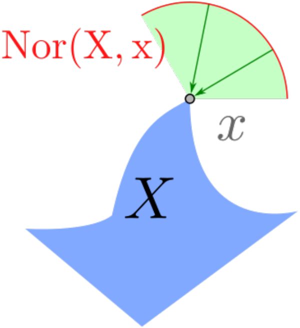
- $\text{reach}_1(X) = \text{reach}(X) > 0$ means that **there is only one closest point in X in a neighborhood of X.**



A Small word on curvatures

Normal Cones of sets with positive reach

Take $X \subset \mathbb{R}^d$ of positive reach. Put $\mathrm{Nor}(X, x)$ set of directions with closest point x in X in a small neighborhood.



Normal Cycles of sets with positive reach

The **normal bundle** of X with $\mathrm{reach}(X) > 0$

$$\mathrm{Nor}(X) = \bigcup_{x \in \partial X} \{x\} \times \mathrm{Nor}(X, x)$$

is a $d-1$ lipschitz submanifold of $\mathbb{R}^d \times \mathbb{S}^{d-1}$.

Integrating over it yields its Normal Cycle N_X.

Joseph Fu's Contribution

The positive reach case

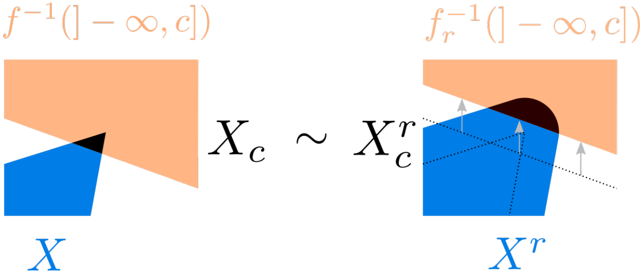
Let $X \subset \mathbb{R}^d$ be of positive reach.

- $x \in X$ is a **critical point** when $\nabla f(x) = 0$ or $-\frac{\nabla f(x)}{\|\nabla f(x)\|} \in \mathrm{Nor}(X, x)$.
- Fu also defines a (more involved) Hessian using properties of the **Normal Cycle**.

Fu's Results

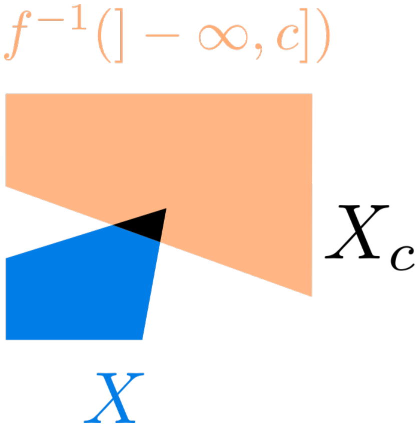
With those new definitions, **the Morse Theorems apply.**

Proof Idea



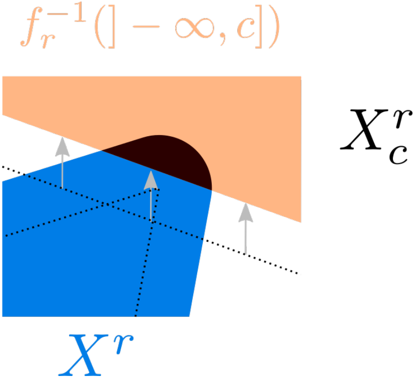
Proof Idea

c regular value $\implies$ X_c **has a positive reach.**



Proof Idea

c regular value of $f_{|X} \implies$ for $r > 0$ small enough, X_c^r **has positive reach "containing"** X_c**.**



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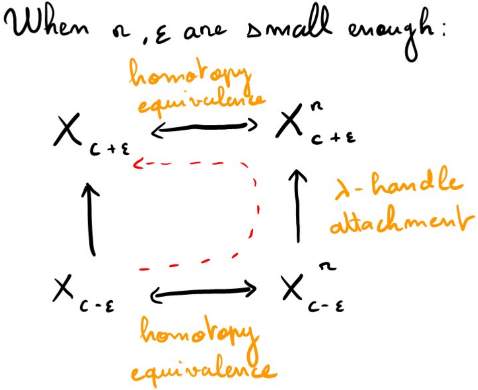

When r, ε are small enough:
homotopy equivalence
$X_{c+\varepsilon}$
$X^r_{c+\varepsilon}$
λ-handle attachment
$X_{c-\varepsilon}$
$X^r_{c-\varepsilon}$
homotopy equivalence

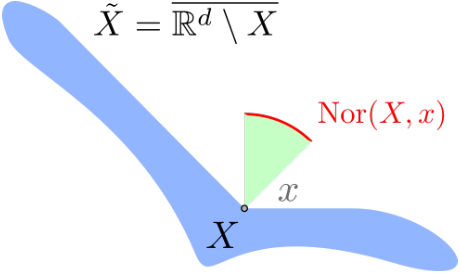
Our Contribution

Settings

$$\text{Put } \tilde{X} = \overline{\mathbb{R}^d \setminus X} \text{ and assume } \operatorname{reach}(\tilde{X}) > 0 \text{ and } \operatorname{reach}_\mu(X) > 0.$$

$$\text{Put } \operatorname{Nor}(X, x) = -\operatorname{Nor}(\tilde{X}, x).$$

$\Longrightarrow$ **Keep the definition of critical points** $-\nabla f(x) \in \operatorname{Cone}(\operatorname{Nor}(X, x))$

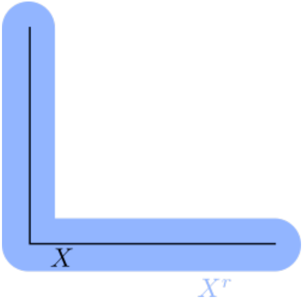


Main result

With those definitions, **the Morse Theorems apply.**

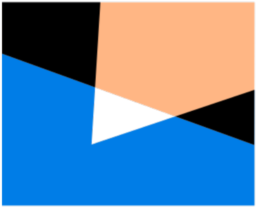
Why those conditions ?

These definitions are verified by any tubular neighborhood A^r when $\mathrm{reach}_\mu(A) > 0$ for r small enough.



Proof Idea - Concept

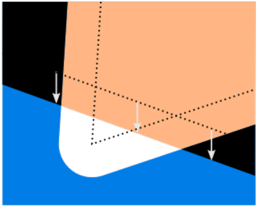
$f^{-1}(]-\infty, c])$



X

$$X_c \sim X_c^{-r}$$

$f_r^{-1}(]-\infty, c])$

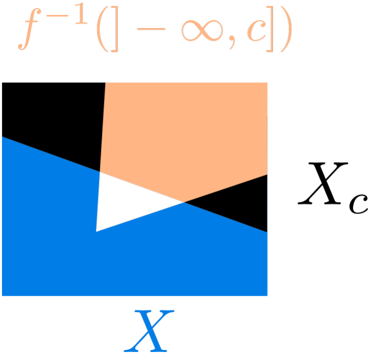


X^{-r}

Proof idea - Technique

$$X_c = \phi_c^{-1}(0) \text{ with } \phi_c = d_X + \max(f - c, 0)$$

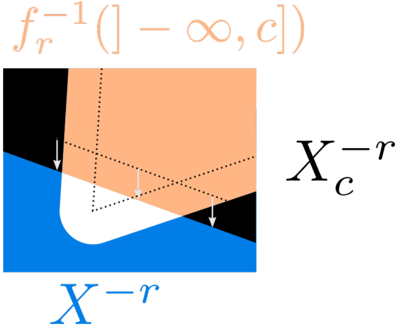
c regular value $\iff$ $\mathrm{sep}(\partial^* \phi_c(x), \{0\}) > 0$ **uniformly in a small neighborhood of X_c.**



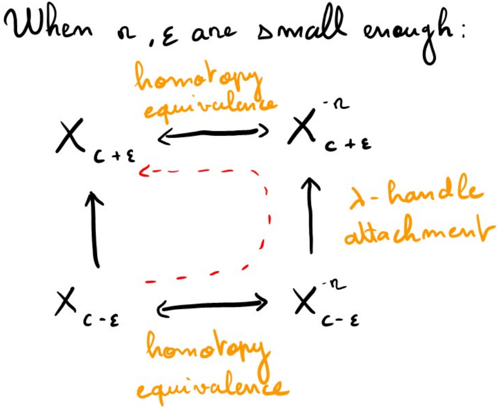
Proof idea - Technique

$$X_c^r = \phi_{c,r}^{-1}(0) \text{ with } \phi_{c,r} = d_{X^{-r}} + \max(f_r - c, 0)$$

c regular value of $f_{|X} \implies \text{sep}(\partial^* \phi_{c,r}(x), \{0\}) > 0$ **uniformly in a small neighborhood of X_c^r containing X_c for $r > 0$ small enough.**



Proof Idea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Thank you !

Thank you for listening !